The problem can be submitted until March 22, 12 :00 noon, either at the exercise session or into the box in front of MA C1 563.

Student(s) [1] :

Question 1 : *The question is worth 5 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 *Reserved for the corrector*

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polyhedron. Two extreme points x_1, x_2 ($x_1 \neq x_2$) of P are said to be *adjacent* if there exists a valid inequality $d^T x \leq \beta$ of P such that

$$P \cap \{x \in \mathbb{R}^n \mid d^T x = \beta\} = \{\lambda x_1 + (1-\lambda)x_2 \mid \lambda \in [0,1]\}$$

Show the following statement : Two different vertices v_1 and v_2 are adjacent, if and only if there exist two bases $B_1, B_2 \subset \{1, \ldots, m\}$ such that $|B_1 \cap B_2| = n-1$ and $v_1 = A_{B_1}^{-1} b_{B_1}$ as well as $v_2 = A_{B_2}^{-1} b_{B_2}$ holds.

Sol. : *L* et $P = \{x \mid Ax \leq b\}$ where $A \in \mathbb{R}^{m \times n}$. We denote the i-th row of A as the row vector a_i. Also for a pair of points x_1, x_2 define $\operatorname{seg}(x_1, x_2) := \{\lambda x_1 + (1-\lambda)x_2 \mid 0 \leq \lambda \leq 1\}$

$\Longleftarrow$ There exists $B \subseteq [m]$ such that

- $r(A_B) = n-1$, and
- $A_B x_1 = b_B$ and $A_B x_2 = b_B$.

Define $d := \sum_{i \in B} a_i$ and $\alpha = \sum_{i \in B} b_i$. Consider the constraint $d^T x \leq \alpha$. Clearly,

- it is a valid inequality for P,
- for each $x \in \operatorname{seg}(x_1, x_2)$, $A_B x = b_B$, and $d^T x = \alpha$.

Let $Z = P \cap \{x \in \mathbb{R}^n \mid d^T x = \beta\}$. We have shown that $\operatorname{seg}(x_1, x_2) \subseteq Z$.

Now consider any $y \in Z$. Then $a_i^T y = b_i$ for each $i \in B$. If $y \notin Z$, then $a_i^T < b_i$ for some $i \in B$ and $d^T y < \alpha$. Thus, $A_B y = b_B$. Since $r(A_B) = n-1$ we have that Z belongs to a line in $\mathbb{R}^n$. Thus, we have shown that Z contains the segment $\operatorname{seg}(x_1, x_2)$ and is contained in some line. This line must be the line joining x_1 and x_2, $L := \{\mu x_1 + (1-\mu)x_2 \mid \mu \in \mathbb{R}\}$. Let $z \in L \setminus \operatorname{seg}(x_1, x_2)$. If $z \in P$ then we can write at least one of x_1 or x_2 as a convex combination of two distinct points in P and this contradicts the fact that x_1 and x_2 are vertices of P. Thus, there can be no such z and $Z = \operatorname{seg}(x_1, x_2)$.

$\Longrightarrow$ Suppose that there is a $d \in \mathbb{R}^n$ such that the $P \cap \{x \in \mathbb{R}^n \mid d^T x = \beta\}$ is exactly $\operatorname{seg}(x_1, x_2)$ for some $x_1 \neq x_2 \in P$. Define

$$K := \{k \in [m] \mid a_k^T x_1 = b_k,\ a_k^T x_2 = b_k\}.$$

1. You are allowed to submit your solutions in groups of at most three students.

Clearly $A_K x = b_K$ for each $x \in \text{seg}(x_1, x_2)$. Since $A_K(x_2 - x_1) = 0$, $\text{r}(A_K) \leq n - 1$. Suppose towards contradiction that $r(A_K) < n - 1$. Then,

$$r(\begin{bmatrix} A_k \\ (x_2 - x_1)^T \\ d^T \end{bmatrix}) \leq n.$$

Thus, $\exists v \neq 0$ such that $a_k^T v = 0$ for each $k \in K$ $(*)$, $d^T v = 0$ and $(x_2 - x_1)^T v = 0$.

Now consider $\bar{x} = (x_1 + x_2)/2$. Of course $\bar{x} \in P$ and $A_K \bar{x} = b_K$. Let $\bar{K} := [m] \setminus K$. For each $k \in \bar{K}$, $a_k^T \bar{x} < b_k$ $(**)$ (here we make use of the fact that $\bar{x} \in \text{seg}(x_1, x_2) \setminus \{x_1, x_2\}$). Thus $A_K \bar{x} = b_K$ and $A_{\bar{K}} \bar{x} < b_{\bar{K}}$. By our choice of v above and by $(*)$ and $(**)$, $\exists \varepsilon > 0$ such that $A(\bar{x} \pm \varepsilon v) \leq b$ and thus $\bar{x} \pm \varepsilon v \in P$. Further, $\bar{x} \pm \varepsilon v \notin \text{seg}(x_1, x_2)$ since $v \perp x_2 - x_1$. However, $d^T(\bar{x} + \varepsilon v) = d^T \bar{x} = \alpha$, contradicting the fact that $P \cap \{x \in \mathbb{R}^n \mid d^T x = \beta\}$ is exactly $\text{seg}(x_1, x_2)$.

To conclude, denote by I_1 all inequalities tight for x_1 and by I_2 all inequalities tight for x_2. Clearly, all rows appearing in A_K appear in both I_1 and I_2. Since $r(A_K) = n - 1$ and x_1 and x_2 are vertices, we can find inequalities $a_1 \in I_1$ and $a_2 \in I_2$, both linearly independent from the rows of A_K such that $A_K \cup \{a_1\}$ is a basis for x_1 and $A_K \cup \{a_2\}$ is a basis for x_2.