École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2019)

Assignment 6

Problem 1

Determine the dual program for the following linear programs:

1.
$$\begin{array}{rcl} \max \quad 2x_1 + 3x_2 - 7x_3 & & \\ x_1 + 3x_2 + 2x_3 & = & 4 \\ x_1 + x_2 & \leq & 8 \\ x_1 - x_3 & \geq & -15 \\ x_1, x_2 & \geq & 0 \end{array}$$

2.
$$\begin{array}{rcr} \min \quad 3x_1 + 2x_2 - 3x_3 + 4x_4 & & \\ 2x_1 - 2x_2 + 3x_3 + 4x_4 & \leq & 3 \\ x_2 + 3x_3 + 4x_4 & \geq & -5 \\ 2x_1 - 3x_2 - 7x_3 - 4x_4 & = & 2 \\ x_1 \geq 0 & & \\ x_4 \leq 0 & & \end{array}$$

Problem 2

In the setting of the matrix-game described in Section 5.1 of the lecture notes, show that for $A \in \mathbb{R}^{m \times n}$, one has
$$\max_i \min_j A(i,j) \leq \min_j \max_i A(i,j).$$

Problem 3

Consider the following linear program $\max\{c^T x : Ax \leq b\}$ and its dual $\min\{b^T y : A^T y = c, y \geq 0\}$. Suppose that both programs are bounded and feasible. Let x_0 and y_0 be feasible solutions of the primal, respectively the dual linear program. Show that the following are equivalent:

(i) x_0 and y_0 are optimal solutions of the primal, respectively the dual.

(ii) $c^T x_0 = b^T y_0$.

(iii) If a component of y_0 is positive, the corresponding inequality in $Ax \leq b$ is satisfied by x_0 with equality.

Problem 4

For each of the following assertions, provide a proof or a counterexample.

(i) An index that has just left the basis B in the simplex algorithm cannot enter in the very next iteration.

(ii) An index that has just entered the basis B in the simplex algorithm cannot leave again in the very next iteration.

Problem 5

We define two different norms on vectors. The infinity-norm is defined by $\|y\|_\infty = \max_i |y_i|$ and the 1-norm is defined by $\|y\|_1 = \sum_i |y_i|$.

Let A be an $m \times n$ matrix and let $b \in \mathbb{R}^m$ be a vector. Consider the problem of minimizing $\|Ax - b\|_\infty$ over all $x \in \mathbb{R}^n$.

Suppose that v is the optimal value of the problem.

(a) Let $p \in \mathbb{R}^m$ be a vector satisfying $\|p_i\|_1 \leq 1$ and $p^T A = 0$. Show that $p^T b \leq v$.

(b) To obtain the best possible lower bound of the form considered in (a), we construct the following linear program

$$\begin{array}{lrcl} \max & p^T b & & \\ & p^T A & = & 0 \\ & \sum_{i=1}^m |p_i| & \leq & 1. \end{array}$$

Using strong duality, show that the optimal solution of this problem is equal to v.

Problem 6

Consider the following problem. We are given $B \in \mathbb{N}$, and a set of integer points

$$S = \{p \in \mathbb{Z}^n : \ 0 \leq p_i \leq B, \ \forall i = 1, \dots, n\},$$

whose points are all colored blue but one, which is red. We have an oracle that, given $i \in \{1, \dots, n\}$ and $\alpha \in \{0, \dots, B\}$, tells us whether there exists a red point $x^* \in S$ with $x_i^* \leq \alpha$. Give an algorithm to find the red point using $O(n \log(B))$ many oracle calls.