École Polytechnique Fédérale de Lausanne
Prof. F. Eisenbrand

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**Discrete Optimization** (Spring 2019)

# Assignment 4

## Problem 1

Consider the polyhedron:

$$P = \begin{cases} x_1 + 2x_2 + x_3 & \leq & 5 \\ 3x_1 + x_2 + x_3 & \leq & 3 \\ x_1 & \leq & 1 \\ x_1 + x_2 & \leq & 2 \\ x_2 + x_3 & \leq & 3 \\ x_1 & \geq & 0 \\ x_1 + x_2 & \geq & 0 \\ x_2 + x_3 & \geq & 0 \end{cases}$$

State which of the following points are vertices of $P$: $p_0 = (0, 0, 3)$, $p_1 = (0, 1, 1), p_2 = (1, 4, -4)$, $p_3 = (1/2, 3/2, 0), p_4 = (1, -1, 1)$.

**Solution:**
For each point $p$, we need to check whether the submatrix of the inequalities that $p$ satisfies with equality has full rank (i.e. equal to 3), and whether $p$ is in $P$. Proceeding this way, we see that only $p_0$ and $p_4$ are vertices.

## Problem 2

Consider the following classification problem: we are given $p_1, \ldots, p_N$ points in $\mathbb{R}^d$, and each point is colored either blue or red. We want to determine if there is an hyperplane $\alpha = \{ax = b\}$ that strictly separates the blue points from the red ones (i.e. such that $ap_i > b$ for all blue points and $ap_i \leq b$ for all red points) and, in case of a positive answer, find such $\alpha$. Show how to solve this problem using linear programming.

**Solution:**
Consider the following linear program (notice that $a \in \mathbb{R}^d$, $b \in \mathbb{R}$ are variables):

$$\begin{array}{rrcll} \max & \epsilon & & & \\ & ap_i & \geq & 1 + \epsilon & \forall\, p_i \text{ blue} \\ & ap_i & \leq & 1 & \forall\, p_i \text{ red} \\ & \epsilon & \geq & 0 & \end{array}$$

If a separating hyperplane exists, then (by changing the sign of the coefficients or by slightly translating it) we can write it as $ax = 1$ and if $\epsilon = \min\{ap_i : p_i \text{ is blue}\}$, we have that $(a, \epsilon)$ is a feasible solution to the linear program with positive objective value. On the other hand, if there is a feasible solution with positive objective value, then the corresponding hyperplane $ax = 1$ strictly separates the blue points from the red.

**Problem 3**

Recall the linear program from the last assignment:

$$\begin{aligned} \max \quad & a + 3b \\ \text{s.t.} \quad & a + b \leq 2 && (1) \\ & a \leq 1 && (2) \\ & -a \leq 0 && (3) \\ & -b \leq 0 && (4) \end{aligned}$$

Solve it with the Simplex method starting with the initial feasible basic solution induced by the constraints (2) and (4). For each iteration indicate the current basis and the corresponding vertex, $\lambda_B$, the direction in which the Simplex moves and how far it moves. At the end indicate the optimal objective value and the proof of optimality (i.e. the final $\lambda$).

**Solution:**

We give the result of the Simplex calculations below:

| iteration | basis | vertex | $\lambda$ | direction | $\epsilon$ | index exchange |
|---|---|---|---|---|---|---|
| 1 | $\{2,4\}$ | $(1,0)^T$ | $(1,-3)^T$ | $(0,1)^T$ | 1 | $4 \Rightarrow 1$ |
| 2 | $\{1,2\}$ | $(1,1)^T$ | $(3,-2)^T$ | $(-1,1)^T$ | 1 | $2 \Rightarrow 3$ |
| 3 | $\{1,3\}$ | $(0,2)^T$ | $(3,2)^T$ | | | |

The last vertex is the optimal solution with an objective function value of 6. It is best to also check this with a drawing.

**Problem 4**

Consider the following linear program:

$$\begin{aligned} \max \quad & 6a + 9b + 2c \\ \text{subject to} \quad & a + 3b + c \leq -4 && (1) \\ & b + c \leq -1 && (2) \\ & 3a + 3b - c \leq 1 && (3) \\ & a \leq 0 && (4) \\ & b \leq 0 && (5) \\ & c \leq 0 && (6) \end{aligned}$$

Solve the linear program with the Simplex method and initial vertex $(-1,-1,0)^T$. For each iteration indicate all the parameters as in the previous exercise including the optimal value and the proof of optimality.

**Solution:**

We give the result of the Simplex calculations below:

| iteration | basis | vertex | $\lambda$ | direction | $\epsilon$ | index exchange |
|---|---|---|---|---|---|---|
| 1 | $\{1,2,6\}$ | $(-1,-1,0)^T$ | $(6,-9,5)^T$ | $(3,-1,0)^T$ | 1/3 | $2 \Rightarrow 4$ |
| 2 | $\{1,4,6\}$ | $(0,-4/3,0)^T$ | $(3,3,-1)^T$ | $(0,1/3,-1)^T$ | 5/2 | $6 \Rightarrow 3$ |
| 3 | $\{1,3,4\}$ | $(0,-1/2,-5/2)^T$ | $(5/2,1/2,2)^T$ | | | |

The last vertex is optimal with an objective function value of $-19/2$.

**Problem 5**

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $A$ has rank equal to $n$. The goal is to show that we can find an initial vertex (or show that $P$ is empty) by solving an appropriate LP:

1. Show that $P$ is non-empty (feasible) if $b \geq 0$.

2. Show that $Q = \{(x, y) \in \mathbb{R}^{n+m} \mid Ax \leq b + y,\, y \geq 0\}$ has a vertex $v$.

3. Create a LP for $Q$ (with initial basic solution corresponding to $v$) that either shows whether $P$ is feasible or not.

**Solution:**
If $b \geq 0$, then the all zero vector $(0, \cdots, 0)^T = 0_n \in \mathbb{R}^n$ is a solution of $Ax \leq b$ and so $P$ is non-empty. This shows that $Q$ is also feasible: setting $y_i \geq \max\{-b_i, 0\}\ \forall i \in [m]$ and $x_i = 0\ \forall i \in [n]$, $(x, y) \in Q$. To show that $Q$ has a vertex, we need to show that the system of inequalities defining $Q$ has rank $n + m$: Defining

$$\hat{A} = \begin{bmatrix} A & -I_m \\ 0 & -I_m \end{bmatrix}$$

$\hat{b} = (b, 0) \in \mathbb{R}^{2m}$, we get that

$$Q = \{z \in \mathbb{R}^{n+m} \mid \hat{A}z \leq \hat{b}\}$$

and $\hat{A}$ has rank $n + m$. This, together with the fact that $Q$ is feasible, shows that there is a vertex (see for instance Problem 3 of Assignment 2). Let us call this vertex $v$. If $v$ has its last $m$ components equal to 0, then (up to leaving away the last $m$ coordinates of $v$) $v \in P$. If not, we would like to use the Simplex method to move to another vertex of $Q$ where the last $m$ coordinates are 0. Define the objective function $c = (0_n, \mathbb{1}_m)^T \in \mathbb{R}^{n+m}$ and consider the following linear program (LP):

$$\min\{c^T z \mid \hat{A}z \leq \hat{b}\}$$

with initial vertex $v$. This LP is clearly bounded below by 0 since the last $m$ entries of any feasible point need to be positive. If the LP returns the value 0, then there is a feasible point of $P$: The vertex corresponding to the objective function value 0 has its last $m$ components equal to zero, so leaving them away, the resulting point belongs to $P$. If the LP returns some value strictly larger than 0, then $P$ is infeasible: The Simplex method finds a point in $Q$ minimizing the objective function and to any feasible point $p \in P$, there corresponds a feasible point $\hat{p} \in Q$ ($\hat{p}$ is obtained by adding $m$ 0's to $p$) and $c^T \hat{p} = 0$. So the LP can only return a value strictly larger than 0 if $P$ is infeasible.