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**Discrete Optimization** (Spring 2019)

# Assignment 3

**Problem 1**

Show the "if" direction of the Farkas' lemma: given $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, if there exists a $\lambda \in \mathbb{R}^m_{\geqslant 0}$ such that $\lambda^\top A = 0$ and $\lambda^\top b = -1$, then the system $Ax \leq b$ is unfeasible.

**Solution:**

Suppose that there exists $x^* \in \mathbb{R}^n$ such that $Ax^* \leq b$. Then, since $\lambda \geq 0$, we have:

$$\lambda^\top Ax^* \leq \lambda^\top b \implies 0 \leq -1,$$

a contradiction.

**Problem 2**

A polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ *contains a line*, if there exists a nonzero $v \in \mathbb{R}^n$ and an $x^* \in \mathbb{R}^n$ such that for all $\lambda \in \mathbb{R}$, the point $x^* + \lambda \cdot v \in P$. Show that a nonempty polyhedron $P$ contains a line if and only if $A$ does not have full column-rank.

**Solution:**

Assume that $P$ contains a line $\{x \in \mathbb{R}^n \colon x = x^* + \lambda \cdot v,\ \lambda \in \mathbb{R}\}$. We claim that $v \in ker(A)$, i.e. for all rows $a_i$ of $A$ we have $a_i^T v = 0$. Assume for contradiction that there is a row $a_i$ with $a_i^T v \neq 0$. Then we can choose $\lambda \in \mathbb{R}$ such that $a_i^T x^* + \lambda a_i^T v > b_i$ (namely such that $|\lambda| > |\frac{b_i - a_i^T x^*}{a_i^T v}|$). Thus for $x := x^* + \lambda v$ we have $x \notin P$ because $a_i^T x > b_i$. This is a contradiction to the fact that $P$ contains the line $\{x \in \mathbb{R}^n \colon x = x^* + \lambda \cdot v,\ \lambda \in \mathbb{R}\}$.

Thus the kernel of $A$ is not empty, and $A$ does not have full column rank.

Conversely, if $A$ does not have full column rank, let $x^*$ be some feasible point of the polyhedron, and let $v$ be a nonzero vector from the kernel of $A$. Then $x^* + \lambda \cdot v \in P$ for all $\lambda \in \mathbb{R}$. Hence $P$ contains a line.

**Problem 3**

Given $x^* = (0\ 1\ 1)^T \in \mathbb{R}^3$ and the vector $d = (1\ 1\ -1)^T \in \mathbb{R}^3$ decide if the ray $\{x^* + \lambda d : \lambda \in \mathbb{R}_{\geq 0}\}$ intersects the following hyperplanes while moving in the direction of $d$. Give the order in which the trajectory passes the planes.

$$P_1 = \{x \in \mathbb{R}^3 : (1\ 2\ 3)x = 0\} \qquad P_2 = \{x \in \mathbb{R}^3 : (3\ 2\ 1)x = 4\}$$
$$P_3 = \{x \in \mathbb{R}^3 : (1\ 1\ 1)x = 2\} \qquad P_4 = \{x \in \mathbb{R}^3 : (0\ 1\ 3)x = -1\}$$

**Solution:**

The trajectory of $x^*$ is given by the line $\{x^* + \delta d : \delta \geq 0\}$ where a point in the trajectory moves further away from $x^*$ if $\delta$ becomes larger. To find the order in which $x^*$ passes the planes we search the corresponding $\delta_i$ for which $x^* + \delta_i d$ is in the plane $P_i$ or decide that such a $\delta$ does not exist.

$P_1$: $(1\ 2\ 3)((0\ 1\ 1)^T + \delta(1\ 1\ -1)^T) = 5 + 0 \cdot \delta = 5 \neq 0$ for all $\delta$, so $x^*$ does not pass $P_1$ since it moves parallel to it.
$P_2$: $(3\ 2\ 1)((0\ 1\ 1)^T + \delta(1\ 1\ -1)^T) = 3 + 4\delta = 4$ for $\delta = \frac{1}{4}$.
$P_3$: $(1\ 1\ 1)((0\ 1\ 1)^T + \delta \cdot (1\ 1\ -1)^T) = 2 + \delta = 2$ for $\delta = 0$, so $x^*$ is already on $P_3$.
$P_4$: $(0\ 1\ 3)((0\ 1\ 1)^T + \delta \cdot (1\ 1\ -1)^T) = 4 - 2\delta = -1$ for $\delta = \frac{5}{2}$.

The order in which $x^*$ passes the planes is $P_3$, $P_2$, $P_4$. The plane $P_1$ will never be passed.

**Problem 4**

Provide a proof or counterexample to the following statement:
Let $\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$ be a linear program with $A \in \mathbb{R}^{m \times n}$ of full column rank. If $B$ is an optimal basis, then all the components of $\lambda_B$ are strictly positive.

**Solution:**
False. Consider the linear program

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 1 \\ & -x_1 \leq 0 \\ & -x_2 \leq 0 \end{aligned}$$

The feasible region of this LP is the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, and $(0,1)$. An optimal basis is $\{1,2\}$ with corresponding feasible basic solution $(0,1)$. Here,

$$\lambda^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

which implies $\lambda^{\mathrm{T}} = [1 \quad 0]$.

**Problem 5**

Consider the following LP:

$$\begin{array}{lrcrcrcrr} \max & 2x & + & 4y & + & 3z & & & \\ \text{s.t.} & 2x & - & 3y & - & z & \leq & 3, & (1) \\ & -x & + & 6y & + & 4z & \leq & 5, & (2) \\ & -x & + & 3y & + & 2z & \leq & 2, & (3) \\ & -x & & & & & \leq & 0, & (4) \\ & & - & y & & & \leq & 0, & (5) \\ & & & & - & z & \leq & 0. & (6) \end{array}$$

a) Given the basis $B = \{1, 2, 6\}$, compute $x^*$ with $A_B x^* = b_B$.

b) Decide whether $x^*$ is feasible.

c) Compute $\lambda \in \mathbb{R}^3$ with $\lambda^T A_B = c^T$.

d) Decide whether $B$ is an optimal basis.

**Solution:**

a) Calculate $A_B^{-1}$ and write $x^* = A_B^{-1} b_B$

$$x^* = \frac{1}{9} \begin{bmatrix} 6 & 3 & 6 \\ 1 & 2 & 7 \\ 0 & 0 & -9 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 33 \\ 13 \\ 0 \end{bmatrix}$$

b) For the feasibility of $x^*$ it is sufficient to see if it fulfills all inequalities not in the basis, i.e. in the set $\{3, 4, 5\}$.

$$\frac{1}{9}(-33 + 3 \cdot 13 + 2 \cdot 0) = \frac{6}{9} \leq 2 \qquad (3)$$

$$\frac{1}{9}(-33 + 0 + 0) \leq 0 \qquad (4)$$

$$\frac{1}{9}(0 - 13 + 0) \leq 0 \qquad (5)$$

c) We reuse $A_B^{-1}$ as calculated in a) to get the equation $\lambda^T = c^T A_B^{-1}$

$$\lambda^T = \begin{bmatrix} 2 & 4 & 3 \end{bmatrix} \frac{1}{9} \begin{bmatrix} 6 & 3 & 6 \\ 1 & 2 & 7 \\ 0 & 0 & -9 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 16 & 14 & 13 \end{bmatrix}$$

d) $B$ is optimal. To see this, extend the $\lambda$ found in c) to a $\lambda' \in \mathbb{R}^6$ by adding zeros to at the lines not in $B$. Now $\lambda' \geq 0$ and it fulfills the equation $\lambda' A = c^T$. By Definition 4.3 of the lecture notes, $B$ is an optimal basis.