École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2019)

Assignment 3

Problem 1

Show the "if" direction of the Farkas' lemma: given $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$, if there exists a $\lambda \in \mathbb{R}^m_{\geqslant 0}$ such that $\lambda^\top A = 0$ and $\lambda^\top b = -1$, then the system $Ax \leq b$ is unfeasible.

Problem 2

A polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ *contains a line*, if there exists a nonzero $v \in \mathbb{R}^n$ and an $x^* \in \mathbb{R}^n$ such that for all $\lambda \in \mathbb{R}$, the point $x^* + \lambda \cdot v \in P$. Show that a nonempty polyhedron P contains a line if and only if A does not have full column-rank.

Problem 3

Given $x^* = (0\ 1\ 1)^T \in \mathbb{R}^3$ and the vector $d = (1\ 1\ -1)^T \in \mathbb{R}^3$ decide if the ray $\{x^* + \lambda d : \lambda \in \mathbb{R}_{\geq 0}\}$ intersects the following hyperplanes while moving in the direction of d. Give the order in which the trajectory passes the planes.

$$P_1 = \{x \in \mathbb{R}^3 : (1\ 2\ 3)x = 0\} \qquad P_2 = \{x \in \mathbb{R}^3 : (3\ 2\ 1)x = 4\}$$
$$P_3 = \{x \in \mathbb{R}^3 : (1\ 1\ 1)x = 2\} \qquad P_4 = \{x \in \mathbb{R}^3 : (0\ 1\ 3)x = -1\}$$

Problem 4

Provide a proof or counterexample to the following statement:

Let $\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$ be a linear program with $A \in \mathbb{R}^{m\times n}$ of full column rank. If B is an optimal basis, then all the components of λ_B are strictly positive.

Problem 5

Consider the following LP:

$$\begin{array}{lrcrcrcrl}
\max & 2x & + & 4y & + & 3z & & & \\
\text{s.t.} & 2x & - & 3y & - & z & \leq & 3, & (1)\\
 & -x & + & 6y & + & 4z & \leq & 5, & (2)\\
 & -x & + & 3y & + & 2z & \leq & 2, & (3)\\
 & -x & & & & & \leq & 0, & (4)\\
 & & - & y & & & \leq & 0, & (5)\\
 & & & & - & z & \leq & 0. & (6)
\end{array}$$

a) Given the basis $B = \{1, 2, 6\}$, compute x^* with $A_B x^* = b_B$.

b) Decide whether x^* is feasible.

c) Compute $\lambda \in \mathbb{R}^3$ with $\lambda^T A_B = c^T$.

d) Decide whether B is an optimal basis.