École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2019)

Assignment 2

Problem 1

Let $K \subseteq \mathbb{R}^n$ and $v \in K$ an extreme point of K. Show that v cannot be written as a convex combination of other points in K.

Solution:
Since v is an "extreme point" there exists an inequality $a^T x \leq \beta$ valid for K such that $\{v\} = K \cap \{x \in \mathbb{R}^n : a^T x = \beta\}$. We prove the statement by contradiction. Without loss of generality assume that v can be written as a convex combination of two points $u, w \in K$, i.e. $v = \lambda u + (1-\lambda)w$. We obtain that

$$\beta = a^T v = a^T(\lambda u + (1-\lambda)w) = \lambda a^T u + (1-\lambda)a^T w < \lambda\beta + (1-\lambda)\beta = \beta,$$

a contradiction. We used that $a^T u < \beta$ since $a^T u \leq \beta$ ($a^T x \leq \beta$ is valid for K and $u \in K$) and $a^T u \neq \beta$ (v is the only point in K satisfying $a^T x \leq \beta$ with equality). Analogously $a^T w < \beta$.

Problem 2

Find a counterexample (and argue why it is one) for Theorem 3.10 when (1) K is convex but not closed, (2) K is not convex but closed

Solution:
For (1), consider the set $K = \{x \in \mathbb{R}^n \mid \|x\|_2 < 1\}$ and $p = e_1 \notin K$. Since for any $\epsilon > 0$, $B(x, \epsilon)$ intersects K, it is clear that we cannot strictly separate p from K.
For (2), consider the set $K = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$ and take $p = 0$.

Problem 3

Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = n$ and $b \in \mathbb{R}^m$. Let $x^* \in P$ and $A'x \leq b'$ be given as in the lecture, i.e., the sub-system of $Ax \leq b$ consisting of inequalities that are satisfied by x^* with equality. Suppose that x^* is not a vertex. We know already that this is equivalent to $\text{rank}(A') < n$. In this exercise, you will show that P contains at least one vertex.

i) Show that there exists a $d \in \mathbb{R}^n$ with $d \neq 0$ and $A'd = 0$.

ii) With this d, show that the line $\{x^* + \lambda d : \lambda \in \mathbb{R}\}$ is not contained in P.

iii) Deduce that there exists a feasible point y^* of P whose sub-system $A''x \leq b''$ of inequalities that are satisfied by y^* with equality, satisfies $\text{rank}(A'') > \text{rank}(A')$.

iv) Conclude that P has a vertex.

Solution:
A' does not have full row rank ($\text{rank}(A') < n$), so we can find a $d \neq 0$ such that $A'd = 0$. If we denote by A'' the rows in A but not in A', we must have that $A''d \neq 0$: If all rows a_r of A'' were orthogonal to d (meaning $a_r d = 0$), then $\text{rank}(A) < n$, contradiction. We call such a row a''. Since $a''d \neq 0$ and $A'd = 0$, we have that the row a'' is linearly independent from the rows in A' (*).

Choosing now λ big enough, we have that $x^* \pm \lambda d$ is not contained in P (where we choose the sign depending on the sign of $a''d$). Conversely, since $A''x^* < b''$ and $A'd = 0$, we can choose $\epsilon > 0$ sufficiently small, such that $A(x^* \pm \epsilon d) \leq b$. For all rows $a_i \in A''$, denote by $\delta_i = b_i - a_i x^* > 0$. We can choose $\epsilon = \min |\delta_i / d_i|$ and we have that $A(x^* \pm \epsilon d) \leq b$ with at least one more row with equality (when taking the appropriate sign). This gives a new subsystem A'_{new} that is satisfied by equality, by (*), $\text{rank}(A'_{new}) > \text{rank}(A')$. To conclude, we redo this procedure until we have found a vertex v such that $\hat{A}v = \hat{b}$ for some subsystem $\hat{A}$ of A of rank equal to n.

Problem 4

Show the following: if $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and the system

$$Ax = b,\ x \geq 0 \tag{1}$$

admits a solution, then there is $\hat{x}$ such that $\hat{x}$ has only m non-zero entries and is also a solution to (1).

Solution:

There are two ways to show this exercise, either by directly using the preceding exercise or by reusing the same idea for this new case. We first show the latter:

We show that if there is a solution with strictly more than m non zero entries, then we can find a solution with at least one non zero entry less: Let x be a solution with $l > m$ non-zero entries. We may suppose these correspond to the first l columns of our matrix. Let $B \in \mathbb{R}^{m \times l}$ be the matrix consisting of the first l columns of A. Since $l \geq m+1$, this matrix B does not have full row rank and so we can find y such that $By = 0$. By adding $n - l$ zeroes to the vector y (so passing from a vector in $\mathbb{R}^{l \times 1}$ to a vector in $\mathbb{R}^{n \times 1}$), we have that

$$Ay = 0$$

By construction, y is only non-zero in the first l entries - exactly where x is strictly positive. So $x^* \pm \epsilon y$ is still a feasible solution for (1) for some small $\epsilon > 0$. Like in the previous exercise, we can take ϵ large enough and with the appropriate sign such that we can eliminate (at least) one non-zero coordinate of y without destroying positivity.

The approach using the preceding exercise: Consider the following (equivalent) system:

$$\begin{bmatrix} A \\ -A \\ -I \end{bmatrix} [x] \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

The new matrix has dimensions $\{(n+2m) \times n\}$ and its rank is n (because of the identity matrix we have attached at the end). By the preceding exercise, there exists a vertex v defined by a subsystem $A'x = b'$ where the rank of A' equals to n. But the rank of the matrix $\begin{bmatrix} A \\ -A \end{bmatrix}$ is smaller or equal to m. This shows that for this vertex v, at least $n - m$ inequalities of $-Ix \leq 0$ are satisfied with equality. This implies that v has at most m non zero entries.

Problem 5

A *conic combination* of vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ is a vector of the form $\lambda_1 v_1 + \cdots + \lambda_n v_n$ with $\lambda_i \in \mathbb{R}_{\geq 0}$ for each i. The set of all conic combinations of the $v_1, \ldots, v_k$ is denoted by $\text{cone}(\{v_1, \ldots, v_n\})$.

Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix and let $a_1, \ldots, a_n \in \mathbb{R}^n$ be the columns of A.

i) Show that $\text{cone}(\{a_1, \ldots, a_n\})$ is the polyhedron $P = \{x \in \mathbb{R}^n \colon A^{-1}x \geq 0\}$.

ii) Show that $\operatorname{cone}(\{a_1, \ldots, a_k\})$ for $k \leq n$ is the set

$$P_k = \{x \in \mathbb{R}^n \colon a_i^{-1} x \geq 0, i = 1, \ldots, k,\ a_i^{-1} x = 0, i = k+1, \ldots, n\},$$

where a_i^{-1} denotes the i-th row of A^{-1}.

Solution:

i) We obtain the following (where $[n]$ denotes the set $\{1, 2, \ldots, n\}$):

$$\begin{aligned}\operatorname{cone}(\{a_1, \ldots, a_n\}) &= \{y = \sum_{i \in [n]} \lambda_i a_i :\ \lambda_i \in \mathbb{R}_{\geq 0}\ \forall i \in [n]\} = \{y = A\lambda :\ \lambda \in \mathbb{R}^n_{\geq 0}\} = \\ &= \{y \in \mathbb{R}^n :\ A^{-1} y = \lambda,\ \lambda \geq 0\} = \{y \in \mathbb{R}^n \colon A^{-1} y \geq 0\}.\end{aligned}$$

ii) Analogously one has:

$$\begin{aligned}\operatorname{cone}(\{a_1, \ldots, a_k\}) &= \{y = A\lambda :\ \lambda \in \mathbb{R}^n_{\geq 0},\ \lambda_i = 0 \text{ for } i > k\} = \\ &= \{y \in \mathbb{R}^n :\ A^{-1} y = \lambda,\ \lambda \geq 0,\ \lambda_i = 0 \text{ for } i > k\} = \\ &= \{y \in \mathbb{R}^n \colon a_i^{-1} x \geq 0, i = 1, \ldots, k,\ a_i^{-1} x = 0, i = k+1, \ldots, n\}.\end{aligned}$$