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Discrete Optimization (Spring 2019)

Assignment 1

Problem 1

Picture yourself in the role of a production manager. There are n production tasks to be executed, and a task j requires p_j working hours to be completed.

You have m employees at your disposal that can each, due to his or her qualifications, work on a subset of the tasks. Denote by S_i the set of jobs that employee i can work on.

A work allocation plan has to ensure that all tasks are completed. You want to create an allocation which is also fair: the maximum number of working hours assigned to an employee is to be minimized. Model this problem as a linear program.

Solution:
We introduce variables x_{ij} to denote the working hours of employee i on task j. With a first constraint we would like to enforce that all tasks are processed, i.e., for every task j the sum of hours that the employees are working on it is at least p_j. Formally, we have $\sum_{i:j\in S_i} x_{ij} \geq p_j$ for all $j = 1,\dots,n$. Furthermore, we note that the working hours cannot be negative, so $x_{ij} \geq 0$ for $i = 1,\dots,m$ and $j \in S_i$. The objective is to minimize the function $\max_{i\in\{1,\dots,m\}} \sum_{j\in S_i} x_{ij}$. However, this is not a linear function. The idea is to introduce a new variable t that stands for the maximum in the objective function. To do so, we add the constraints $t \geq \sum_{j\in S_i} x_{ij}$ for each $i = 1,\dots,m$.
Eventually, we obtain the linear program:

$$\begin{aligned}
\min \quad & t & \\
\text{s.t.} \quad & t \geq \sum_{j\in S_i} x_{ij} & i = 1,\dots,m \\
& \sum_{i:j\in S_i} x_{ij} \geq p_j & j = 1,\dots,n \\
& x_{ij} \geq 0 & i = 1,\dots,m, \quad j \in S_i
\end{aligned}$$

This linear program can easily be translated into the inequality standard form.

Problem 2

Consider the problem

$$\begin{aligned}
\min \quad & 2x + 3|y-10| \\
\text{subject to} \quad & |x+2| + y \leq 5,
\end{aligned} \tag{1}$$

and reformulate it as a linear programming problem.

Solution:
To get a linear program, we need to get rid of the absolute values. We will ensure that our linear program has the following properties.

- Every solution of (1) yields a solution of the linear program with the same objective value.

- Each solution of the linear program yields a solution for (1) with at most that objective value.

If our linear program has these two properties this implies that optimal solutions for the linear program induce optimal solutions for (1) and we are done.
The basic idea is to replace the absolute values with a new variable and two constraints as follows: We replace $|y-10|$ with a variable z and the constraints $z \geq y-10$ and $z \geq -y+10$. That means we obtain

$$\begin{aligned} \min \quad & 2x+3z \\ \text{subject to} \quad & |x+2|+y \leq 5 \\ & z \geq y-10 \\ & z \geq -y+10. \end{aligned}$$

Note that $z \geq |y-10|$ in any feasible solution. One can easily check that the two properties from above are fulfilled. Observe that it is crucial at this point that we consider a *minimization* problem. This replacement would not work with a *maximization* problem.
Similarly, we could replace the term $|x+2|$ with a new variable and two additional constraints. However, here there is also an alternative. We can replace

$$|x+2|+y \leq 5 \qquad \Leftrightarrow \qquad \begin{aligned} x+2+y &\leq 5 \quad \text{and} \\ -(x+2)+y &\leq 5 \end{aligned}$$

This gives

$$\begin{aligned} \min \quad & 2x+3v \\ \text{subject to} \quad & x+2+y \leq 5 \\ & -(x+2)+y \leq 5 \\ & v \geq y-10 \\ & v \geq -y+10. \end{aligned}$$

Again one can easily check that both properties are fulfilled.

Problem 3

Prove the following statement or give a counterexample: The set of optimal solutions of a linear program is always finite.

Solution:
A counterexample $\max\{0 \cdot x \text{ s.t. } 0 \leq x \leq 1\}$.

Problem 4

Consider the following linear program:

$$\begin{array}{rrcrcl} \max & x & + & y & & \\ \text{s.t.} & 3x & + & 2y & \leq & 6 \\ & x & + & 4y & \leq & 4. \end{array}$$

The solution $(x, y) = (8/5, 3/5)$ satisfies the both constraints and has the objective value $11/5$. Provide a certificate that this is an optimal solution.

Solution:
The desired certificate is $(3/10, 1/10)$. Summing up the two constraints above with multipliers $3/10$ and $1/10$ respectively, one obtains that any feasible solution has to satisfy the inequality

$$x+y \leq 11/5. \tag{2}$$

Observe that (2) is at the same time an upper bound on the objective function value. Furthermore, the solution $(x, y) = (8/5, 3/5)$ satisfies this constraint with equality, thus it is optimal.

Problem 5

Let (3) be a linear program in inequality standard form, i.e.

$$\max\{c^{\mathrm{T}}x \mid Ax \le b, x \in \mathbb{R}^n\}, \tag{3}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.
Prove that there is an equivalent linear program (4) of the form [1]

$$\min\{\tilde{c}^{\mathrm{T}}x \mid \tilde{A}x = \tilde{b}, x \ge 0, x \in \mathbb{R}^{\tilde{n}}\}, \tag{4}$$

where $\tilde{A} \in \mathbb{R}^{\tilde{m}\times\tilde{n}}$, $\tilde{b} \in \mathbb{R}^{\tilde{m}}$, and $\tilde{c} \in \mathbb{R}^{\tilde{n}}$ are such that every optimal point of (3) corresponds to an optimal point of (4) and vice versa.

Solution:
The transformation requires three steps:

1. Replace every variable x_j with two non-negative variables $x_j^+ := \max\{x_j, 0\}$ and $x_j^- := -\min\{x_j, 0\}$, and replace every occurrence of x_j with $(x_j^+ - x_j^-)$.
2. Replace every constraint of the form $a_{i1}x_1 + \ldots + a_{in}x_n \le b_i$ with a constraint $a_{i1}x_1 + \ldots + a_{in} + s_i = b_i$, where s_i is a new, non-negative *slack* variable.
3. Multiply the objective function with -1 to obtain a minimization problem.

Combining these three steps, we can write the transformed linear program as

$$\begin{aligned}
\min \quad & -c^T x^+ + c^T x^- + 0^T s \\
\text{subject to} \quad & Ax^+ - Ax^- + s = b \\
& x^+ \ge 0 \\
& x^- \ge 0 \\
& s \ge 0
\end{aligned}$$

This is the desired form if we set $\tilde{c} = \begin{pmatrix}-c & c & 0\end{pmatrix}^T$ and $\tilde{A} = \begin{bmatrix}A & -A & I\end{bmatrix}$, where I denotes the $m \times m$ identity matrix.
Given a feasible solution x of the original linear program, we can find a feasible solution $\tilde{x} = \begin{pmatrix}x^+ & x^- & s\end{pmatrix}^T$ of the reformulated program by setting x^+ to the positive part of x, x^- to the negative part of x as above, and $s = b - Ax$. Note that $s \ge 0$ holds, since x is feasible, i.e. x satisfies $Ax \le b$. Furthermore, $\tilde{A}\tilde{x} = Ax + b - Ax = b$ by construction. Thus $\tilde{x}$ is feasible and we have $\tilde{c}^T\tilde{x} = -c^T x$.
Conversely, given a feasible solution $\tilde{x} = \begin{pmatrix}x^+ & x^- & s\end{pmatrix}^T$ of the reformulated program, $x = x^+ - x^-$ defines a feasible solution of the original linear program. We have $Ax \le \tilde{A}\tilde{x} = b$, since $s \ge 0$ holds by the feasibility of $\tilde{x}$. Note that x has objective function value $c^T x = -\tilde{c}^T\tilde{x}$.
This implies in particular that optimal solutions correspond to each other.

[1] This is called *equality standard form*.