École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2019)

Assignment 2

Problem 1

A *convex combination* of the points $v_1, \dots, v_k \in \mathbb{R}^n$ is a point of the form $\lambda_1 v_1 + \dots + \lambda_n v_n$ where $\lambda_i \geq 0$ for each i and $\lambda_1 + \dots + \lambda_n = 1$.

Let $K \subseteq \mathbb{R}^n$ and $v \in K$ an extreme point of K. Show that v cannot be written as a convex combination of other points in K.

Problem 2

Find a counterexample (and argue why it is one) for Theorem 3.10 when (1) K is convex but not closed, (2) K is not convex but closed.

Problem 3

Consider a polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ with $A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = n$ and $b \in \mathbb{R}^m$. Let $x^* \in P$ and $A'x \leq b'$ be given as in the lecture, i.e., the sub-system of $Ax \leq b$ consisting of inequalities that are satisfied by x^* with equality. Suppose that x^* is not a vertex. We know already that this is equivalent to $\operatorname{rank}(A') < n$. In this exercise, you will show that P contains at least one vertex.

i) Show that there exists a $d \in \mathbb{R}^n$ with $d \neq 0$ and $A'd = 0$.

ii) With this d, show that the line $\{x^* + \lambda d \colon \lambda \in \mathbb{R}\}$ is not contained in P.

iii) Deduce that there exists a feasible point y^* of P whose sub-system $A''x \leq b''$ of inequalities that are satisfied by y^* with equality, satisfies $\operatorname{rank}(A'') > \operatorname{rank}(A')$.

iv) Conclude that P has a vertex.

Problem 4

Show the following: If $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and the system

$$Ax = b,\, x \geq 0$$

admits a solution, then there exists a solution $\hat{x}$ that has only m non-zero entries.
Hint: Use the previous exercise.

Problem 5

A *conic combination* of vectors $v_1, \dots, v_k \in \mathbb{R}^n$ is a vector of the form $\lambda_1 v_1 + \dots + \lambda_n v_n$ with $\lambda_i \in \mathbb{R}_{\geq 0}$ for each i. The set of all conic combinations of the $v_1, \dots, v_k$ is denoted by $\operatorname{cone}(\{a_1, \dots, a_n\})$.

Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix and let $a_1, \dots, a_n \in \mathbb{R}^n$ be the columns of A.

i) Show that $\operatorname{cone}(\{a_1, \dots, a_n\})$ is the polyhedron $P = \{y \in \mathbb{R}^n \colon A^{-1}y \geq 0\}$.

ii) Show that $\operatorname{cone}(\{a_1, \dots, a_k\})$ for $k \leq n$ is the set

$$P_k = \{y \in \mathbb{R}^n \colon a_i^{-1}x \geq 0, i = 1, \dots, k,\, a_i^{-1}x = 0, i = k+1, \dots, n\},$$

where a_i^{-1} denotes the i-th row of A^{-1}.