Subset sum and Knapsack problem using Fourier analysis (Master project)

1 Introduction

Consider the following innocent looking problem: Given are n integers, $x_1, \cdots, x_n$ and a target T. Decide whether there exists a subset $S \subset \{1, ..., n\}$ such that $\sum_{i \in S} x_i = T$. This problem is known as the Subset Sum problem.
There are essentially two easy ways how to solve it: either by a brute force approach with running time $O(2^{n/2})$ or with dynamic programming in time $O(nT)$ (there is reason to believe one cannot do better than $O(T)$). In this project, we will concentrate on the dynamic programming approach: using the Fast Fourier Transform (FFT), one can do better than $O(nT)$.
A generalization of this problem is the Knapsack problem: in addition to the Subset problem, each x_i has a cost c_i and we are interested in the cheapest way to reach the target T: find S such that $\sum_{i \in S} x_i = T$ and $\sum_{i \in S} c_i$ minimal.

2 Aim of the project

In a first part, the goal is to understand the theory behind the FFT. Then we will consider two algorithms using the FFT, counting the number of solutions to the subset sum problem in time approximately $O(T \ln(n))$ and applications to Knapsack problems, [1], and solving subset sum in time $O(\sqrt{n}T)$, [2]. For better understanding, these algorithms should be implemented and be compared to the running time of a naive algorithm.
Depending on how the project advances, we can consider a third article that improves the running time of subset sum to $O^*(T + n)$ (with high probability and neglecting some logarithmic factors), [3]. This article also relies on the FFT technique but also on some facts from number theory and on some probabilistic construction. Alternatively, we could also work on (min, +) convolution that can be used to improve the running time for Knapsack type problems, [4].

3 Prerequisites

Strong interest in Optimization and Combinatorics and willingness to refresh 1st year programming skills.

References

[1] Yu. Nesterov, *Fast Fourier Transform and its Applications to Integer Knapsack Problems*, CORE Discussion Paper No. 2004/64

[2] K. Koiliaris C. Xu, *A Faster Pseudopolynomial Time Algorithm for Subset Sum*, available at https://arxiv.org/abs/1507.02318

[3] K. Bringmann, *A Near-Linear Pseudopolynomial Time Algorithm for Subset Sum*, available at https://arxiv.org/abs/1610.04712

[4] D. Bremner et al., *Necklaces, Convolutions, and X+Y*, available at https://arxiv.org/pdf/1212.4771.pdf