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Due Date: May 06, 2010
Discussions: March 15, April 22, April 29

Discrete Optimization

Spring 2010

Solutions 4

You can hand in written solutions for up to two of the exercises marked with (*) or (Δ) to obtain bonus points. The duedate for this is May 06, 2010, before the exercise session starts. Math students are restricted to exercises marked with (*). Non-math students can choose between (*) and (Δ) exercises.

Exercise 1 (Δ)
Let $M \in \mathbb{Z}^{n\times m}$ be totally unimodular. Prove that the following matrices are totally unimodular as well:

1. M^T
2. $(M \quad I_n)$
3. $(M \quad -M)$
4. $M \cdot (I_n - 2e_j e_j^T)$ for some j

I_n is the $n \times n$ identity matrix, and e_j is the vector having a 1 in the jth component, and 0 in the other components.

Solution

1. Let A be a square submatrix of M^T. Then
$$\det(A) = \det(A^T) \in \{-1, 0, 1\}$$
as A^T is a square submatrix of M and M is totally unimodular.

2. Let A be a square submatrix of $(M \quad I_n)$. Let $a_1, \ldots, a_k$ be the columns of A that originate from I_n. Hence, each of these columns has as most one 1-entry, the other entries are 0. Hence, using Laplace-expansion successively along these columns we get that $|\det(A)| = |\det(A')|$ for some square submatrix of M. Since M is TU, this shows $\det(A) \in \{-1, 0, 1\}$.

3. Let A be a square submatrix of $(M \quad -M)$. Let $a_1,\dots,a_k$ be the columns of A that originate from $-M$. Let A' be the matrix obtained from A by multiplying $a_1,\dots,a_k$ with -1. Hence
$$|\det(A)| = |\det(A')|.$$
Now we distinguish two cases.

Case 1: A' is (up to permutation of columns) a square submatrix of M. Since M is TU, we have $\det(A') \in \{-1,0,1\}$.

Case 2: A' has at least two identical columns. Hence $\det(A') = 0$.

We conclude that in both cases we have $\det(A) \in \{-1,0,1\}$.

4. Observe that $M \cdot (I_n - 2e_j e_j^T)$ is obtained from M by multiplying one column with -1. Thus, $M \cdot (I_n - 2e_j e_j^T)$ is (up to permutation of columns) a submatrix of $(M \quad -M)$. Since $(M \quad -M)$ is TU, so is $M \cdot (I_n - 2e_j e_j^T)$.

Exercise 2 (*)

A family $\mathscr{F}$ of subsets of a finite groundset E is *laminar*, if for all $C, D \in \mathscr{F}$, one of the following holds:
$$(i)\ C \cap D = \emptyset,\ \ (ii)\ C \subseteq D,\ \ (iii)\ D \subseteq C.$$

Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two laminar families of the same groundset E and consider its union $\mathscr{F}_1 \cup \mathscr{F}_2$. Define the $|\mathscr{F}_1 \cup \mathscr{F}_2| \times |E|$ adjacency matrix A as follows: For $F \in \mathscr{F}_1 \cup \mathscr{F}_2$ and $e \in E$ we have $A_{F,e} = 1$, if $e \in F$ and $A_{F,e} = 0$ otherwise.

Show that A is totally unimodular.

Solution

Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two laminar families on the same groundset E, and A the corresponding adjacency matrix. Observe that *every* square submatrix of A also is the adjacency matrix of two laminar families: Removing a row from A corresponds to deleting a set from the laminar families. Removing a column from A corresponds to removing an element of the groundset from all sets of the laminar families. Both operations preserve the structure of laminar families.

For that reason, it is sufficient to show the following statement: Every square matrix A that is adjacency matrix of two laminar families has determinant ± 1 or 0.

We will transform A with elementary column operations as follows: Let $e \in E$ be an element from the groundset that is contained in at least two sets from $\mathscr{F}_1$. Let $S_1,\dots,S_k$ be the sets of $\mathscr{F}_1$ with $e \in S_i$. Using the properties of laminar families, we know that there is a $l \in \{1,\dots,k\}$ such that $S_l \subseteq S_i$ for all $i = 1,\dots,k$. Redefine $S_i := S_i \setminus S_l$ for all $i \neq l$. Observe that $\mathscr{F}_1$ is still a laminar family after this modification. Also observe that the operation of removing the set S_l corresponds to subtraction the row S_l from the other rows S_i in the matrix A.

Hence we can apply this transformation until each $e \in E$ is contained in at most one set of $\mathscr{F}_1$. Similarly we can apply this transformation to $\mathscr{F}_2$ until each $e \in E$ is contained in at most one set of $\mathscr{F}_2$. Applying the corresponding elementary row operations to A yields a matrix A' with $\det(A) = \det(A')$. A' has the property that there are two disjoint subsets of rows, the rows corresponding to $\mathscr{F}_1$ and the rows corresponding to $\mathscr{F}_2$, such that each column of A' has at most one 1-entry in the rows of $\mathscr{F}_1$ and at most one 1-entry in the rows of $\mathscr{F}_2$. All other entries are 0.

Let A'' be the submatrix of A' consisting only of the columns with two 1-entries. Note that this is a node-edge incidence matrix of a bipartite graph. Hence A'' is TU as seen in the lecture. With Exercise 1.2 we get that A' is TU. Hence $\det(A) \in \{-1, 0, 1\}$.

Exercise 3

Consider the following scheduling problem: Given n tasks with periods $p_1, \ldots, p_n \in \mathbb{N}$, we want to find offsets $x_i \in \mathbb{N}_0$, such that every task i can be executed periodically at times $x_i + p_i \cdot k$ for all $k \in \mathbb{N}_0$. In other words, for all pairs i, j of tasks we require $x_i + k \cdot p_i \neq x_j + l \cdot p_j$ for all $k, l \in \mathbb{N}_0$.

Formulate the problem of finding these offsets as an integer program (with zero objective function).

Solution

Fix two tasks i and j. We rewrite the feasibility condition

$$x_i + k \cdot p_i \neq x_j + l \cdot p_j \quad \forall k, l \in \mathbb{N}_0$$

as

$$x_i - x_j \neq l \cdot p_j - k \cdot p_i \quad \forall k, l \in \mathbb{N}_0,$$

and see that this is equivalent to

$$x_i - x_j \neq l \cdot p_j - k \cdot p_i \quad \forall k, l \in \mathbb{Z}.$$

For an integer $a \in \mathbb{N}$, let $a \cdot \mathbb{Z} := \{ax : x \in \mathbb{Z}\}$. Thus we can write the feasibility condition more compact as

$$x_i - x_j \notin p_j \cdot \mathbb{Z} + p_i \cdot \mathbb{Z}.$$

Recall from your algebra course that $p_j \cdot \mathbb{Z} + p_i \cdot \mathbb{Z} = \gcd(p_i, p_j)\mathbb{Z}$. Hence the feasibility criterion is equivalent to

$$x_i - x_j \notin \gcd(p_i, p_j) \cdot \mathbb{Z}.$$

This can be expressed with the linear constraint

$$1 \leq x_i - x_j + z_{ij} \gcd(p_i, p_j) \leq \gcd(p_i, p_j) - 1,$$

where z_{ij} is an integer variable.

Hence, we get the following integer program:

$$\min\left\{0 : 1 \leq x_i - x_j + z_{ij} \gcd(p_i, p_j) \leq \gcd(p_i, p_j) - 1 \;\; \forall \text{tasks } i, j, i \neq j, \;\; z_{ij} \in Z, x_i \in \mathbb{N}_0\right\}.$$

Exercise 4 (*)
Let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ be a polyhedron. Show that the following are equivalent for a feasible x^*:

i) x^* is a vertex of P.

ii) There exists a set $B \subseteq \{1, \ldots, m\}$ such that $|B| = n$, A_B is invertible and $A_B x^* = b_B$. Here the matrix A_B and the vector b_B consists of the rows of A indexed by B and the components of b indexed by B respectively.

iii) For every feasible $x_1, x_2 \in P$, $x_1 \neq x^* \neq x_2$, one has $x^* \notin \text{conv}\{x_1, x_2\}$.

Solution

(ii)⇒(i): Note that x^* is the vertex of the roof A_B. Let $c \in \text{cone}(A_B)$ be a *strictly positive* conic combination of the rows of A_B. As seen in Exercise 5 on Sheet 2, the vertex x^* of the roof A_B is then a unique optimal solution to the linear program $\max\{c^T x \,:\, A_B x \leq b_B\}$. Since $x^* \in P$, this implies that x^* is the unique optimal solution to the linear program $\beta := \max\{c^T x \,:\, Ax \leq b\}$. Hence, $c^T x \leq \beta$ for all $x \in P$ and $\{x \in \mathbb{R}^n \,:\, c^T x = \beta\} \cap P = x^*$. Thus x^* is a vertex.
(i)⇒(iii): Let $H := \{x \in \mathbb{R}^n \,:\, c^T x = \beta\}$ be the hyperplane that defines the vertex x^*, i.e. $H \cap P = \{x^*\}$, and $c^T x \leq \beta$ for all $x \in P$. Assume there are $x_1, x_2 \in P$, $x_1 \neq x^* \neq x_2$ such that $x^* = \lambda x_1 + (1-\lambda) x_2$ for some $\lambda \in (0,1)$. We have

$$\beta = c^T x^* = \lambda \underbrace{(c^T x_1)}_{<\beta,\text{ as } x_1 \notin H} + (1-\lambda) \cdot \underbrace{(c^T x_2)}_{<\beta,\text{ as } x_2 \notin H} < \beta,$$

a contradiction.
(iii)⇒(ii): Let $A_=$ and $b_=$ be the rows of the system $Ax \leq b$ that are satisfied with equality by x^*, i.e.

$$A_= x^* = b_=.$$

We claim that $A_=$ has full column rank, i.e. the set $S := \{x \in \mathbb{R}^n \,:\, A_= x = b_=\}$ has unique solution x^*.

Assume for contradiction that $A_=$ does not have full rank. Then S contains a line $L = \{x \in \mathbb{R}^n \,:\, x = x^* + \lambda v, \lambda \in \mathbb{R}\}$ for some $v \in \mathbb{R}^n$. Since $a_i x^* < b_i$ for all rows of $A \setminus A_=$, there is an $\varepsilon > 0$ such that $x_1 := x^* + \varepsilon v \in P$ and $x_2 := x^* - \varepsilon v \in P$. Note that $x^* \in \text{conv}\{x_1, x_2\}$, in contradiction to the assumption.

Thus $A_=$ is of full column rank and (ii) follows.

Exercise 5
Show the following: A polyhedron $P \subseteq \mathbb{R}^n$ with vertices is integral, if and only if each vertex is integral.

Solution
If P is integral, by definition each vertex is integral.

Now assume that $P = \{x \in \mathbb{R}^n : Ax \le b\}$ is a polyhedron with vertices and assume that each vertex is integral. To show that P is integral we show that each face of P contains a vertex. Let x^* be a vertex of P.

Now consider some face F defined by the hyperplane $H = \{x \in \mathbb{R}^n : c^T x = \beta\}$, i.e. $H \cap P = F$ and $c^T x \le \beta$ for all $x \in P$. By construction, the set of optimal solutions for the linear program

$$\max\{c^T x : Ax \le b\} \tag{1}$$

is F.

Since x^* is a vertex of P, with Exercise 4 it follows that A is of full column rank. Hence we can apply the simplex algorithm to solve (1). The simplex algorithm will compute an optimal solution y that is the vertex of a roof. Again by Exercise 4, we get that y is a vertex of P, thus integral. Since y is optimal, by construction we have $y \in F$. Thus the face F contains an integer point.

Exercise 6

Consider the polyhedron $P = \{x \in \mathbb{R}^3 : x_1 + 2x_2 + 4x_3 \le 4,\ x \ge 0\}$. Show that this polyhedron is integral.

Solution

We can write P as follows:

$$P = \{x \in \mathbb{R}^3 : Ax \le b\}, \tag{2}$$

where $A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $b = (4,\ 0,\ 0,\ 0)^T$. Clearly A is of full column rank, thus with Exercise 4 we get that P has a vertex. Using Exercise 5 we know that it is sufficient to show that every vertex is integer. Again using the characterization from Exercise 4, we know that the vertices of P are defined as the unique solutions of the $\binom{4}{3}$ many subsystems obtained from (2) by choosing three out of the four rows: The unique solution of the system

$$\begin{pmatrix} 1 & 2 & 4 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} x = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

is $x = (0,0,1)^T$. The unique solution of the system

$$\begin{pmatrix} 1 & 2 & 4 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

is $x = (0,2,0)^T$. The unique solution of the system

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

is $x = (4,0,0)^T$. The unique solution of the system

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is $x = (0,0,0)^T$. Hence, P is integral.

Exercise 7
Which of these matrices is totally unimodular? Justify your answer.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Solution
The first matrix is not TU, as its determinant is 2.

The second matrix is TU. This is because it is an interval matrix, and one can show that every interval matrix is TU. Here, we will prove TUness of this matrix directly. One can easily check that the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ is TU. Applying Exercise 1.2, this implies that $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is TU. Hence, the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is TU because duplicating rows preserves TUness. For the same reason, this implies that $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ is TU. Finally again with exercise 1.2, we get that $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$ is TU and we are done.

Exercise 8
Consider the complete graph G_n with 3 vertices, i.e., $G = (\{1,2,3\}, \binom{3}{2})$. Is the polyhedron of the linear programming relaxation of the vertex-cover integer program integral?

Solution
The vertex-cover IP for the G_3 looks as follows:

$$\begin{aligned} \min \quad & w_1 x_1 + w_2 x_2 + w_3 x_3 & \\ & x_1 + x_2 & \geq 1 \\ & x_1 + x_3 & \geq 1 \\ & x_2 + x_3 & \geq 1 \\ & x_1, x_2, x_3 \in \mathbb{N}_0 & \end{aligned}$$

Let $w_1 = w_2 = w_3 = 1$. Observe that we need at least two of the vertices in our vertex cover, i.e. the optimum of this integer program is 2. On the other hand, the vector $x = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$ is a feasible solution to the linear programming relaxation of value $\frac{3}{2} < 2$.

Assume that the polyhedron of the LP relaxation is integral. Thus, with Lemma 4.25 from the lecture, the simplex algorithm will compute an optimal integer solution to the relaxation. As we have seen above, every integer solution has a value of at least 2, while an optimum fractional solution has a value of $\frac{3}{2}$, a contradiction.