Prof. Friedrich Eisenbrand
Martin Niemeier

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Discussions: March 25, April 01

Discrete Optimization

Spring 2010

Solutions 3

You can hand in written solutions for up to two of the exercises marked with (*) or (Δ) to obtain bonus points. The duedate for this is April 15, 2010, before the exercise session starts. Math students are restricted to exercises marked with (*). Non-math students can choose between (*) and (Δ) exercises.

Exercise 1 (*)

Prove the following statement:

Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax = b$, $x \geq 0$ has a solution if and only if for all $\lambda \in \mathbb{R}^m$ with $\lambda^T A \geq 0$ one has $\lambda^T b \geq 0$.

Hint: Use duality theory!

Solution

The statement we want to prove is known as Farkas' Lemma. Let $c = (0,\dots,0)^T \in \mathbb{R}^n$, and consider the linear program

$$\min\{c^T x \,:\, Ax = b,\ x \geq 0\} \tag{1}$$

and its dual

$$\max\{b^T y \,:\, A^T y \leq 0\}.$$

Set $\lambda := -y$. We get the equivalent LP

$$\max\{-b^T \lambda \,:\, A^T \lambda \geq 0\}. \tag{2}$$

Assume that $Ax = b$, $x \geq 0$ has a solution. Thus (1) is feasible. Since $c = 0$, is is also bounded as every feasible solution has the same objective value. By duality theory this implies that (2) is feasible and bounded and

$$0 = \min\{c^T x \,:\, Ax = b,\ x \geq 0\} = \max\{-b^T \lambda \,:\, A^T \lambda \geq 0\}.$$

Hence for each λ with $A^T \lambda \geq 0$ we have $-b^T \lambda \leq 0$. This shows the first part of the claim.

For the converse, assume that $\lambda^T b \geq 0$ for all λ with $\lambda^T A \geq 0$. Observe that the linear program (2) is feasible, as $\lambda = 0$ is a solution. Since $\lambda^T b \geq 0$ for all feasible solutions, the linear program is bounded. By duality theory, its dual (2) is feasible which asserts the claim.

Exercise 2

1. Consider the linear program

$$\max\{c^T x \,:\, Ax \le b\}$$

with $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$. The dual of the linear program is

$$\min\{b^T y \,:\, A^T y = c,\; y \ge 0\}.$$

Let x and y be feasible solutions to the primal and dual LP, respectively. Prove the following statement:

The vectors x and y are optimal solutions to their respective LPs if and only if for each $i = 1,\dots,m$ we have $a_i^T x = b_i$ or $y_i = 0$.
Hint: Consider the inner product $(b - Ax)^T y$ and show that it is 0 if and only if x and y are optimal. Why is this sufficient to show?

2. Consider the following linear program:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \le 6 \\ & x_1 + 2x_2 \le 8 \\ & 3x_1 + 4x_2 \le 22 \\ & x_1 + 5x_2 \le 23 \end{aligned}$$

Show that $(4/3, 10/3)$ is an optimal solution by computing an optimal solution of the dual using the first part of this exercise.

Solution

1. The statement we want to prove is the so called *complementary slackness theorem*. Consider the inner product

$$(b - Ax)^T y. \tag{3}$$

As x is a feasible solution to the primal LP, we have $b - Ax \ge 0$. Because y is a feasible solution to the dual LP, we have $y \ge 0$.

Thus (3) is 0 if and only if for each $i = 1,\dots,m$ we have that $b - a_i^T x = 0$ or $y_i = 0$. Hence it is sufficient to show that (3) is 0 if and only if x and y are optimal solutions.

Observe that

$$(b - Ax)^T y = b^T y - x^T A^T y = b^T y - x^T c.$$

The second equality used that fat that y is a feasible solution to the dual LP. Using duality theory, we know that x and y are optimal if and only if $b^T y = c^T x$. In other words, (3) is 0 if and only if x and y are optimal solutions.

2. The dual LP is the following:

$$\begin{array}{rl} \min & 6y_1 + 8y_2 + 22y_3 + 23y_4 \\ \text{subject to} & 2y_1 + y_2 + 3y_3 + y_4 = 1 \\ & y_1 + 2y_2 + 4y_3 + 5y_4 = 1 \\ & y_1, y_2, y_3, y_4 \geq 0 \end{array}$$

Observe that the solution $(4/3, 10/3)$ satisfies the first two constraints of the primal LP with equality, while the last two constraints are satisfied with strict inequality. Applying the complementary slackness theorem to this situation, we get that *if* x is optimal, then *every* optimal solution y to the dual has $y_3 = y_4 = 0$.

Thus the two constraints of the dual simplify to

$$\begin{aligned} 2y_1 + y_2 &= 1 \\ y_1 + 2y_2 &= 1. \end{aligned}$$

The system has full rank, thus there is a unique solution: $y_1 = y_2 = \frac{1}{3} \geq 0$. Thus $y = (\frac{1}{3}, \frac{1}{3}, 0, 0)$ is feasible. The complementary slackness theorem tells us that x and y are optimal for their respective LPs.

We check this by computing the objective values: $6y_1 + 8y_2 + 22y_3 + 23y_4 = \frac{14}{3} = x_1 + x_2$. Weak duality asserts that x and y are optimal.

Exercise 3

Consider the linear program

$$\max\{c^T x \,:\, Ax \leq b\}$$

with $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$. The dual is

$$\min\{b^T y \,:\, A^T y = c,\ y \geq 0\}.$$

The following table describes nine situations for the status of primal and dual LPs. Decide which one of these situations are possible/impossible. If they are possible, give an example. Otherwise give a short argument why they are impossible.

		Dual		
		Finite optimum	Unbounded	Infeasible
Primal	Finite optimum			
	Unbounded			
	Infeasible			

Solution

- Primal and dual feasible and bounded is *possible*: Example is $c = b = (0)$ and $A = (0)$.
- Primal feasible and bounded, dual unbounded is *impossible*: Assume that $Ax \leq b$ is has a solution x. Then by weak duality, $c^T x$ a lower bound for all solutions to the dual, in contradiction to the fact that the dual is unbounded.
- Primal feasible and bounded, dual infeasible is *impossible*: If the primal has an optimal solution, the duality theorem tells us that the dual has an optimal solution as well. In particular the dual is feasible.
- Primal unbounded and dual feasible and bounded is *impossible*: Assume that $A^T y = c$ has a solution y. Then by weak duality, $b^T y$ is an upper bound for all solutions to the primal, in contradiction to the fact that the primal is unbounded.
- Primal unbounded, dual unbounded is *impossible*: As seen above, if the primal is unbounded, then the dual is infeasible.
- Primal unbounded, dual infeasible is *possible*: Example is $c = (1)$, $b = (0)$ and $A = (0)$.
- Primal infeasible, dual feasible and bounded is *impossible*: With the strong duality theorem, if the dual is feasible and bounded, so is the primal.
- Primal infeasible, dual unbounded is *possible*: Example is $c = (0)$, $b = (-1)$ and $A = (0)$.
- Primal and dual infeasible is *possible*: Example is $c = (1)$, $b = (-1)$ and $A = (0)$.

Exercise 4
Suppose you are given an oracle algorithm, which for a given polyhedron

$$P = \{\tilde{x} \in \mathbb{R}^{\tilde{n}} \ : \ \tilde{A}\tilde{x} \leq \tilde{b}\}$$

gives you a feasible solution or asserts that there is none. Show that using a *single* call of this oracle one can obtain an *optimum* solution for the LP

$$\max\{c^T x \ : \ x \in \mathbb{R}^n; \ Ax \leq b\},$$

assuming that the LP is feasible and bounded.
Hint: Use duality theory!

Solution
The LP is feasible and bounded, thus an optimum solution must exist. Strong duality tells us that the dual $\min\{b^T y : A^T y = c, \ y \geq 0\}$ is feasible and bounded. For optimal solutions x^* of the primal and y^* of the dual we have $b^T y^* = c^T x^*$.

Thus every point (x^*, y^*) of the polyhedron

$$\begin{aligned} c^T x &= b^T y \\ Ax &\leq b \\ A^T y &= c \\ y &\geq 0 \end{aligned}$$

is optimal. Hence with one oracle call for the polyhedron above we get an optimal solution of the LP.

Exercise 5 (Δ)
A *directed graph* $D = (V, A)$ is a tuple consisting of a set of *vertices* V and a set of *arcs* $A \subseteq V \times V$. Given an arc $a = (u, v) \in A$, the vertex u is called the *tail* of a and v is called the *head* of a. For some vertex $v \in V$, the set of arcs whose tail is v is called

$$\delta^{out}(v) := \{a \in A \,:\, a = (v, u) \text{ for some } u \in V\}.$$

Analogous, the set of arcs whose head is v is called

$$\delta^{in}(v) := \{a \in A \,:\, a = (u, v) \text{ for some } u \in V\}.$$

Let $s, t \in V$ and let $c : A \to \mathbb{N}_0$ be a capacity function. A function $f : A \to \mathbb{N}_0$ is an $s-t$-*flow*, if for every vertex $v \in V \setminus \{s, t\}$ we have *flow conservation*, i.e.

$$\sum_{a \in \delta^{in}(v)} f(a) - \sum_{a \in \delta^{out}(v)} f(a) = 0.$$

Moreover, $0 \leq f(a) \leq c(a)$ must hold for each $a \in A$.

The *value* of a flow f is defined as $\sum_{a \in \delta^{in}(t)} f(a) - \sum_{a \in \delta^{out}(t)} f(a)$. Formulate the problem of finding an $s-t$-flow of maximum value as a linear program.
Hint: Use the so called node-arc incidence matrix. It has a row for each vertex and a column for each arc. Each column consists of zeros, except for the entries corresponding to the head and the tail of the arc: The head gets a (1) *entry and the tail gets a* (-1) *entry.*

Solution
Let A be the node arc incidence matrix of the graph D, with the columns of s and t removed. We claim that the max flow LP is:

$$\max \sum_{a \in \delta^{in}(t)} x_a - \sum_{a \in \delta^{out}(t)} x_a \tag{4}$$

$$\text{subject to} Ax = 0 \tag{5}$$

$$x \leq c \tag{6}$$

$$x \geq 0. \tag{7}$$

Every solution to this LP is a flow: The vector x encodes the flow on each arc of the graph.

The constraints (5) ensure that for every node $v \in V$, $v \neq s, t$, the incoming flow matches the outgoing flow. I.e. these constraints ensure flow conservation. The constraints (6) and (7) ensure that the capacity constraints and nonnegativity is satisfied.

Finally, the objective function encodes the value of the flow. Hence an optimal solution to the LP corresponds to an optimal flow in the graph.

Exercise 6 (*)

In the lecture you have seen the simplex algorithm for linear programs of the form

$$\max\{c^T x \,:\, Ax \leq b\}.$$

We will now derive a simplex algorithm for linear programs of the form

$$\min\{c^T x \,:\, Ax = b,\ x \geq 0\} \tag{8}$$

with $c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Throughout the exercise we assume that (8) is feasible and bounded, and that A has full row rank.

For $i \in \{1, \ldots, n\}$ we define A_i as the i-th column of A. Moreover, for some subset $B \subseteq \{1, \ldots, n\}$, A_B is the matrix A restricted to the columns corresponding to elements of B.

A subset $B \subseteq \{1, \ldots, n\}$ with $|B| = n$ such that A_B has full rank is called a *basis*. The vector $x \in \mathbb{R}^n$ defined as $x_i := 0$ for all $i \notin B$ and $x_B := A_B^{-1} b$ is called the *basic solution* associated to B. Note that x is a feasible solution to (8) if and only if $x \geq 0$.

Given a basis B and let $j \in \{1, \ldots, n\}$, $j \notin B$. The vector $d \in \mathbb{R}^n$ defined as $d_j = 1$, $d_i = 0$ for all $i \notin B$ and $d_B := -A_B^{-1} A_j$ is called the j*-th basic direction.*

Assume that the solution x associated to B is feasible. Moreover assume that $x_B > 0$.

1. Show that there is a $\theta > 0$ such that $x + \theta d$ is a feasible solution. Give a formula to compute the largest θ such that $x + \theta d$ is feasible.

2. Let θ^* be maximal. Show that there is a basis B' such that $x + \theta^* d$ is the basic solution associated to B'.

3. Let $x' = x + \theta d$. Show that the objective value of x' changes by $\theta\left(c_j - c_B^T A_B^{-1} A_j\right)$.

4. Consider a basis B with basic feasible solution x. Show that if $c - c_B^T A_B^{-1} A \geq 0$, then x is an optimal solution to (8).

This suggests the following algorithm: Start with some basis B whose associated basic solution is feasible. Compute $\bar{c} := c - c_B^T A_B^{-1} A$. If $\bar{c} \geq 0$, we have an optimal solution (see 4). Otherwise, let j be such that $\bar{c}_j < 0$. Part 2 and 3 show that if we change the basis, we find a feasible solution with an improved objective value. We repeat these steps until the vector $\bar{c}$ is nonnegative.

This is the way the simplex algorithm usually is introduced in the literature. This algorithm is exactly the same as the one you learned in the lecture. To get an intuition why this is true, show the following:

5. Given a basis B, show that its associated basic solution is feasible if and only if B is a *roof* of the LP dual to (8).
6. Consider a basis B and its associated feasible basic solution x. As seen before, B is also a roof in the dual LP. Let y be the vertex of that roof.

   Show that for any $j \in \{1,\dots,n\}$ we have $\bar{c}_j < 0$ if and only if $A_j^T y > c_j$.

Solution

The dual of (8) is

$$\max\{b^T y \,:\, A^T y \le c\}. \tag{9}$$

1. Observe that $Ad = 0$ holds. Thus for any θ and $x' := x + \theta d$ we have $Ax' = Ax + \theta A d = b + 0 = b$. Thus x' is a feasible solution if and only if $x' \ge 0$.

   Let $J := \{i \in \{1,\dots,n\} \,:\, d_i < 0\}$. Observe that by construction $J \subseteq B$. If $J = \emptyset$, then $x + \theta d \ge 0$ for all $\theta \ge 0$. Thus x' is feasible for every $\theta \ge 0$ and a maximal θ does not exist.

   If $J \ne \emptyset$, then $x' \ge 0$ if and only if $x_i + \theta d_i \ge 0$ for all $i \in J$. We conclude that

$$\theta^* := \min\{-\frac{x_i}{d_i} \,:\, i \in J\}$$

   is maximal.

2. Let i^* be such that $\theta^* = -\frac{x_{i^*}}{d_{i^*}}$.

   Let $B' := (B \setminus \{i^*\}) \cup \{j\}$. $A_{B'}$ has full rank because the old basis vector A_{i^*} can be written as linear combination of B': $0 = Ad$ implies $A_{i^*} = -\frac{1}{d_{i^*}} A_{B'} d_{B'}$. Let $x' := x + \theta^* d$.

   Observe that $x'_{i^*} = x_{i^*} - \frac{x_{i^*}}{d_{i^*}} d_{i^*} = 0$ and hence $b = Ax' = A_{B'} x'_{B'}$. This shows that x' is the basic solution associated to B'.

3. Observe that

$$c^T x' - c^T x = c^T x + \theta c^T d - c^T x = \theta(c_j d_j - c_B^T A_B^{-1} A_j) = \theta\left(c_j - c_B^T A_B^{-1} A_j\right).$$

4. We prove the claim by giving a dual solution of the same objective value.

   We have that $c - c_B^T A_B^{-1} A \ge 0$, which we can rewrite to

$$A^T (A_B^{-T} c_B) \le c.$$

   Thus the vector $y := A_B^{-T} c_B \in \mathbb{R}^m$ is a feasible solution for the dual (9). Observe that

$$b^T y = b^T A_B^{-T} c_B = c_B^T (A_B^{-1} b) = c_B^T x_B = c^T x.$$

   The weak duality theorem asserts that x is optimal.

5. Since B is a basis, A_B has full rank, and so has A_B^T. Thus the first two properties for a roof are satisfied. We know from the lecture that B is a roof of (9) if and only if the objective vector b can be written as a conic combination of the rows in A_B^T.

   Let x be the basic solution associated to B. Thus we have $A_B x_B = b$. In other words, x_B gives the coefficients of the unique linear combination of b using the rows of A_B^T. Observe that this linear combination is a conic combination if and only if $x \geq 0$.

6. The vertex of the roof B is given as the (unique) solution to the system

$$A_B^T y = c_B.$$

   Thus $y = A_B^{-T} c_B$

   Now fix some j. We have

$$\bar{c}_j := c_j - c_B^T A_B A_j = c_j - A_j^T (A_B^{-T} c_B).$$

   Thus the dual constraint $A_j^T y \leq c_j$ is violated if and only if $\bar{c}_j < 0$.