École Polytechnique Fédérale de Lausanne
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October 27, 2011

Randomized Algorithms (Fall 2011)

Assignment 3

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Note: The purpose of these notes is to give a sketch of <u>one</u> possible solution. We do not guarantee correctness, nor completeness. It is your task to find and report mistakes.

Solution to Problem 1:

We use the algorithm of Raghavan and Tompson. Let $X_e^i = \begin{cases} 1, \text{if chose } s_i\text{-}t_i \text{ path uses } e \\ 0, \text{otherwise} \end{cases}$, $X_e = \sum_{i=1}^k X_e^k$.

Then, as before, $\mathbb{E}[X_e^i] = \sum_{p \in \mathbb{P}_i : e \in p} x_p^i$ and $\mathbb{E}[X_e] = \sum_{i=1}^k \sum_{p \in \mathbb{P}_i : e \in p} x_p^i \leq C$ due to the LP. Since X_e is the sum of independent Poisson trails we can use a Chernoff bound. Note, that for $\mathbb{E}[X_e] \leq C$ we can derive the following bound (the proof is analog to the proof of the original Chernoff bound):

$$\Pr(X \geq (1+\delta)C) \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}} \right)^C .$$

We want this probability to be bounded by $1/(2n^2)$ to later apply the union bound over all edges. Hence, we want to guarantee that

$$\frac{e^\delta}{(1+\delta)^{1+\delta}} \leq \left(2n^2\right)^{-1/C} .$$

This is equivalent to

$$(1+\delta)\ln(1+\delta) - \delta \geq \frac{1}{C} (\ln 2 + 2\ln n) .$$

Setting $\delta = 3\frac{\ln n}{\ln\ln n} - 1$ gives

$$3\frac{\ln n}{\ln\ln n}\ln\ln n - 3\frac{\ln n}{\ln\ln n}\ln\frac{\ln\ln n}{3} - 3\frac{\ln n}{\ln\ln n} = \ln n \left(3 - \frac{3}{\ln\ln n}\left(1 + \ln\frac{\ln\ln n}{3}\right)\right) \geq \ln 2 + 2\ln n \geq \frac{1}{C}(\ln 2 + 2\ln n)$$

for sufficiently large n and assuming $C \geq 1$.
Now, using the union bound, we obtain

$$\Pr(\exists e : X_e > 3\frac{\ln n}{\ln\ln n} C) \leq \frac{1}{2} .$$

Solution to Problem 2:
The algorithm is due to Raghavan and Tompson. This is the same algorithm that was presented in class. In the analysis, after choosing $R \geq 6C$ where C is the optimal congestion (determined by binary search) we obtain $2n^2 \leq 2^R$ which gives $R \in O(\log n)$. Now, the statement that was proved is $\Pr(\exists e : X_e \geq R) \leq \frac{1}{2}$ where X_e is the random variable for the congestion of e. Thus, with probability at most $1/2$ the congestion is greater than $R \in O(\log n)$. Since we know, that the congestion is $\Omega(\log n)$, repeating the algorithm and taking the best solution of the repetitions yields a solution that is only a constant factor away from the optimum (with high probability). Note that this exercise requires a little bit of care with the constants in the O-notation, since the constant in R can potentially be very small.

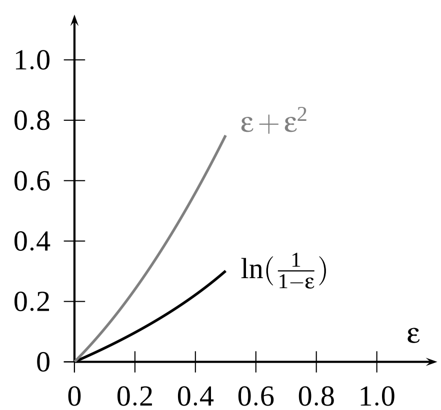
Solution to Problem 3:
The solution to this problem is given in Section 6.3 (Theorem 6.16) of the book "The design of approximation algorithms" by David Williamson and David Shmoys (free online version available at http://www.designofapproxalgs.com/).

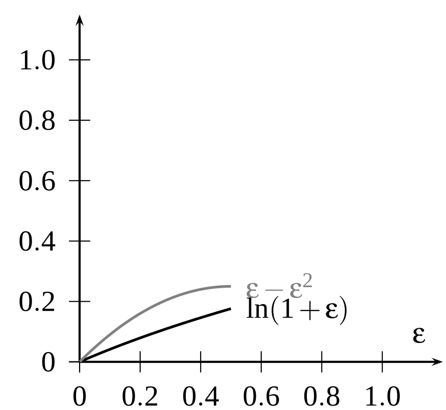
Solution to Problem 4:

1. Let $\varepsilon > 0$, $0 \leq x \leq 1$. Define
$$f(\varepsilon) = (1-\varepsilon x) - (1-\varepsilon)^x$$
then it suffices to show that $f(\varepsilon) \geq 0$ for all $\varepsilon > 0$. Note that $f(0) = 0$. If there would be an $\varepsilon_0 > 0$ with $f(\varepsilon_0) < 0$, then by the *Mean value Theorem* there would be a $\varepsilon_1 > 0$ with $f'(\varepsilon_1) < 0$. Hence we will show that $f'(\varepsilon) \geq 0$ for all $\varepsilon \geq 0$. Then
$$f'(\varepsilon) = -x + x \cdot (1-\varepsilon)^{x-1} = \underbrace{x}_{\geq 0} \cdot (\underbrace{(1-\varepsilon)^{x-1}}_{\geq 1} - 1) \geq 0$$

2. Similar to (1).

4. The claim is equivalent to showing $1+\varepsilon \geq e^{\varepsilon-\varepsilon^2}$. Recall that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$. Hence
$$= e^{\varepsilon-\varepsilon^2} \overset{\text{Tailor series}}{=} \sum_{i=0}^{\infty} \frac{(\varepsilon-\varepsilon^2)^i}{i!} \leq 1 + (\varepsilon-\varepsilon^2) + \sum_{i=2}^{\infty} \frac{\varepsilon^i}{i!} \leq 1 + (\varepsilon-\varepsilon^2) + \frac{1}{2}\varepsilon^2 \underbrace{\sum_{i=0}^{\infty} (1/2)^i}_{=2} = 1+\varepsilon$$





3. Similar to (4).

Solution to Problem 5:
Let $\overline{\ell^t} = \frac{1}{\rho}\ell^t$ be the scaled loss vector. Then $0 \leq \overline{\ell^t_j} \leq 1$, hence the result from the lecture implies that $E[\overline{L}] \leq \frac{\ln(N)}{\varepsilon} + (1+\varepsilon)\overline{L}^j$. I.e.
$$E[L] \leq \rho \frac{\ln(N)}{\varepsilon} + (1+\varepsilon) \underbrace{\rho \overline{L}^j}_{=L^j}$$

Solution to Problem 6:
When setting $w_j^0 := p_j$, the total inital weight is 1 instead of N. Furthermore the weight of expert j at the end is $p_j \cdot (1-\varepsilon)^{L_j}$ hence
$$E[L] \leq \frac{1}{\varepsilon} \cdot \ln\left(\frac{W^0}{p_j(1-\varepsilon)^{L_j}}\right) = \frac{\ln(1/p_j)}{\varepsilon} + L_j \underbrace{\frac{1}{\varepsilon}\ln(\frac{1}{1-\varepsilon})}_{\leq 1+\varepsilon} \leq \frac{\ln(1/p_j)}{\varepsilon} + (1+\varepsilon) \cdot L^j.$$

Next, choose probability distribution $p_j = \frac{6}{\pi^2 \cdot j^2}$. Then indeed $\sum_{j=1}^{\infty} p_j = 1$ and

$$E[L] \leq \frac{\ln(1/p_j)}{\varepsilon} + (1+\varepsilon) \cdot L^j = \frac{\ln(\frac{\pi^2 j^2}{6})}{\varepsilon} + (1+\varepsilon) \cdot L^j \leq \frac{2\ln(j) + 1/2}{\varepsilon} + (1+\varepsilon) \cdot L^j$$