Combinatorial Optimization

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General remark:
In order to obtain a bonus for the final grading, you may hand in written solutions to the exercises marked with a star at the beginning of the exercise session on November 29.

Exercise 1

Let $M = (E, \mathcal{I})$ be an independence system (i.e. $(M0)$ and $(M1)$ hold).
Prove or give a counterexample that the following two properties are equivalent:

$(M2)$ For all $I, J \in \mathcal{I}$ and $|J| > |I|$ there exists some $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$.

$(M2')$ For all $I, J \in \mathcal{I}$ with $I \cap J = \emptyset$ and $|J| > |I|$, there exists some $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$.

Solution

Consider the following counterexample:
Let $E = \{a, b, c, d\}$ and

$$\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}\}$$

$(E, \mathcal{I})$ is clearly not a matroid, since $\{b, d\}$ can not be augmented by an element from $\{a, b, c\}$. However, $(M2')$ clearly holds for every possible combination of sets of size 1 and 2, but because of disjointness, the only combination to consider the set $\{a, b, c\}$ is together with $\{d\}$. In this case, (M2') trivially holds as well.

Exercise 2

Let $G = (V, E)$ be a graph. Show that $\text{conv}(\{\chi_F \in \{0, 1\}^{|E|} \colon F \text{ is a forest in } G\})$, the convex hull of incidence vectors of all forests in G is equal to the following set

$$\{x \in \mathbb{R}_{\geq 0}^{|E|} \colon \sum_{e \in E[T]} x(e) \leq |T| - 1, \text{ for all } \emptyset \neq T \subseteq V\}$$

Solution

This is a nice application of the matroid polytope. We know that $E = E(G)$ with $\mathcal{I} := \{F \subseteq E \colon F \text{ is forest in } G\}$ is a matroid. The corresponding rank function

determines the maximum size of a forest in a subgraph of G. Formally, we have that $r(A) = |T| - k$ for all $A \subseteq E$ where $T \subseteq V$ such that $E[T] = A$ and k is the number of connected components in (T, A).
We have that

$$\text{conv}(\{\chi_F \in \{0,1\}^{|E|} \colon F \text{ is a forest in } G\}) = \{x \in \mathbb{R}^{|E|}_{\geq 0} \colon \sum_{e \in A} x(e) \leq r(A), \text{ for all } A \subseteq E\}$$

"$\subseteq$": Let x fulfill the conditions of the matroid polytope and let $\emptyset \neq T \subseteq V$ be a subset of the nodes of G. We have that $\sum_{e \in E[T]} x(e) \leq r(E[T]) \leq |T| - 1$, since the number of connected components is at least 1.
"$\supseteq$": Let $x \in \mathbb{R}^{|E|}_{\geq 0}$ fulfill $\sum_{e \in E[T]} x(e) \leq |T| - 1$, for all $\emptyset \neq T \subseteq V$. Consider a set $\emptyset \neq A \subseteq E$ (trivial if $\emptyset = A$) and let $T \subset V$ be such that $E[T] = A$. Suppose now that (T, A) has k connected components $T_1, \ldots, T_k$. Then we obtain

$$\sum_{e \in A} x(e) = \sum_{i=1}^{k} \sum_{e \in E[T_i]} x(e) \leq \sum_{i=1}^{k} |T_i| - 1 = |T| - k = r(A)$$

Exercise 3 ($\star$)

Let E be a finite set and let $r : 2^E \to \mathbb{Z}_+$. Then r is the rank function of a matroid $(E, \mathcal{I})$ if and only if for all $I, J \subseteq E$:

(i) $r(I) \leq r(J) \leq |J|$ if $I \subseteq J$,
(ii) $r(I \cap J) + r(I \cup J) \leq r(I) + r(J)$.

Hint: "$\Leftarrow$" Show first that $(E, \mathcal{I})$ where $\mathcal{I} = \{I \subseteq E \colon |I| = r(I)\}$ is a matroid (using condition (iii) in exercise 2 on sheet 4). Conclude that r is indeed a rank function by induction on the size of $A \subseteq E$.

Solution

see Schrijver B, theorem 39.8

Exercise 4

We consider the following generalization of the matroid polytope:
Let E be a finite set and let $f : 2^E \to \mathbb{R}_+$ be a submodular function, i.e. $f(I \cap J) + f(I \cup J) \le f(I) + f(J)$ for all $I, J \subseteq E$. The **polymatroid** defined by f is the polytope

$$P(f) := \left\{ x \in \mathbb{R}^E : x \ge 0, \sum_{e \in A} x_e \le f(A) \qquad \forall A \subseteq E \right\}$$

Note that the matroid polytope is a special case where f is the rank function of a matroid over E. What is the separation problem for a polymatroid?

Let $G = (V, E)$ be a graph and $c : E \to \mathbb{R}_+$. Consider the function $f(A) := c(\delta(A)) = \sum_{e \in \delta(A)} c(e)$ for all $A \subseteq V$.

(i) Show that f is submodular.
(ii) Show how to solve the separation problem for the polymatroid defined by f over V in time $O(m^2 n^2)$.
(iii) Conclude how to maximize a weight function $w : V \to \mathbb{R}$ over the polymatroid $P(f)$.

Solution

The separation problem for a polymatroid is the problem of minimizing the corresponding submodular function.

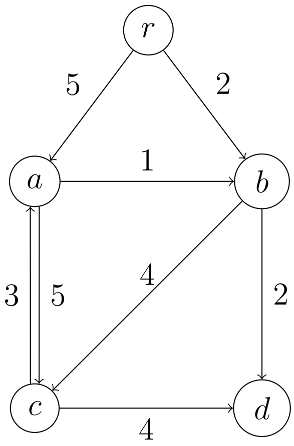
(i) Let $I, J \subseteq V$. We have

$$f(I)+f(J) = c(\delta(I))+c(\delta(J)) = c(\delta(I \cap J))+c(\delta(I \cap J))+c(E[I \setminus J, J \setminus I]) \ge f(I \cap J)+f(I \cup J)$$

(ii) The separation problem is to find a minimum cut in an undirected graph. This can be done by reducing the problem to finding the minimum s-t-cut in a directed graph. Simply replace each edge by a pair of arcs with opposite directions. Then, select a node $s \in V$ as a source and compute for every other node $t \in V \setminus \{s\}$ the minimum s-t-cut. Take the minimum over all t.
This strategy needs at most $n - 1$ calls of Edmonds-Karp, which runs in $O(m^2 n)$.
(iii) binary search together with the ellipsoid method.

Remark: We can optimize over any polymatroid using a generalized greedy method. This implies as well that we can minimize any submodular function.

Exercise 5
Trace the steps of algorithm from the lecture to compute a minimum weight arborescence rooted at r in the following example.



Solution

Minimum weight of an r-arborescence is 11.

Exercise 6 ($\star$)
Let $D = (V, A)$ be a directed graph with root $r \in V$. Suppose that D does not contain an arborescence rooted at r. Prove that there exists a (nonempty) strongly connected component K in D such that $r \notin K$ and $|\delta^{in}(K)| = 0$.

Solution

Prove the contrapositive statement:
If for every strongly connected component K in D, it holds $r \in K$ or $|\delta^{in}(K)| \geq 1$ then D contains an arborescence rooted at r.
We know that D contains an r-arborescence iff every node is reachable from r. However, every node in a connected component is reachable from any other node and by assumption, we have that every strongly connected component that does not contain the root can be reached from another strongly connected component. Since we cannot reenter a strongly connected component, we will eventually end up in a component that contains the root. Thus D contains an r-arborescence.