Prof. F. Eisenbrand EPFL - DISOPT

Combinatorial Optimization

Adrian Bock Fall 2011

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General remark:
In order to obtain a bonus for the final grading, you may hand in written solutions to the exercises marked with a star at the beginning of the exercise session on November 15.

Exercise 1 ($\star$)

In this exercise, we show that solving the separation problem for the perfect matching polytope is sufficient in order to solve the separation problem for the matching polytope.

Let $G = (V, E)$ be a graph and $G' = (V', E')$ a copy of it. Consider the graph $\tilde{G} = (\tilde{V}, \tilde{E})$ where $\tilde{V} := V \cup V'$ and $\tilde{E} = E \cup E' \cup \{\{v, v'\} : v \in V, v' \text{ its copy}\}$.

Let $x \in \mathbb{R}^{|E|}$ be a vector and define $\tilde{x} \in \mathbb{R}^{|\tilde{E}|}$ as $\tilde{x}(e) = \begin{cases} x(e) & \text{if } e \in E \cup E' \\ 1 - x(\delta(v)) & \text{if } e = \{v, v'\} \end{cases}$

Show the following:

(i) $\tilde{x}(e) \geq 0$ for all $e \in \tilde{E}$ if and only if $\sum_{e \in \delta(v)} x(e) \leq 1$ for all $v \in V$ and $x(e) \geq 0$ for all $e \in E$.

(ii) Every $\tilde{y} \in \mathbb{R}^{|\tilde{E}|}$ feasible in the perfect matching polytope of $\tilde{G}$ corresponds to a feasible $y \in \mathbb{R}^{|E|}$ in the matching polytope of G.

(iii) If $\tilde{x}(e) \geq 0$ for all $e \in \tilde{E}$, we have $\tilde{x}(\tilde{\delta}(\tilde{U})) \geq \tilde{x}(\tilde{\delta}(W \setminus X)) + \tilde{x}(\tilde{\delta}(X' \setminus W'))$ for each $\tilde{U} \subseteq \tilde{V}$, where $W = \tilde{U} \cap V$, $X' = \tilde{U} \cap V'$ and W', X are their copies in V', V respectively.

(iv) If $\tilde{x} \geq 0$ and there exists $\tilde{U} \subseteq \tilde{V}$ with $|\tilde{U}|$ odd and $\tilde{x}(\tilde{\delta}(\tilde{U})) < 1$, then there exists $U \subseteq V$ with $|U|$ odd and $\sum_{e \in E[U]} x(e) > \frac{|U|-1}{2}$

Solution

(i) The statement follows directly from the definitions.
If $\tilde{x} \geq 0$ then $x(e) = \tilde{x}(e) \geq 0$ for all $e \in E$ and
$\sum_{e\in\delta(v)} x(e) = 1 - \tilde{x}(\{v, v'\}) \leq 1$ for all $v \in V$. On the other hand,
$\tilde{x}(e) = x(e) \geq 0$ for all $e \in E \cup E'$ and
$\tilde{x}(\{v, v'\}) = 1 - \sum_{e\in\delta(v)} x(e) \geq 0$

(ii) Let $\tilde{y}$ be feasible in the perfect matching polytope of $\tilde{G}$. We set $x(e) := \tilde{y}(e)$ for all $e \in E$ and obtain that $x(e) = \tilde{y}(e) \geq 0$ and $\sum_{e\in\delta(v)} x(e) \leq \sum_{e\in\tilde{\delta}(v)} \tilde{y}(e) = 1$ for all $v \in V$. Finally,

$$\sum_{e\in E[U]} x(e) = \sum_{e\in\tilde{E}[U]} \tilde{y}(e) = 1/2 \left(\sum_{v\in U} \sum_{e\in\tilde{\delta}(v)} \tilde{y}(e) - \sum_{e\in\tilde{\delta}(U)} \tilde{y}(e) \right) \leq \frac{|U| - 1}{2}$$

(iii) Notice that the only edges between W and X' in $\tilde{G}$ are between $W \cap X$ and $W' \cap X'$. We obtain

$$\begin{aligned} \tilde{x}(\tilde{\delta}(U)) = \tilde{x}(\tilde{\delta}(W \cup X')) &= \tilde{x}(\tilde{\delta}(W \setminus X)) + \tilde{x}(\tilde{\delta}(X' \setminus W')) \\ &\quad + x(\delta(W \cap X)) - 2x(E[W \cap X, W \setminus X]) \\ &\quad + x(\delta'(W' \cap X')) - 2x(E[X' \cap W', X' \setminus W']) \\ &\geq \tilde{x}(\tilde{\delta}(W \setminus X)) + \tilde{x}(\tilde{\delta}(X' \setminus W')) \end{aligned}$$

since $x(\delta(W \cap X)) \geq x(E[W \cap X, W \setminus X]) + x(E[W \cap X, X \setminus W])$ and note that x is the same with respect to E and E'.

(iv) From (iii), we obtain (w.l.o.g. $|W \setminus X|$ is odd)

$$\begin{aligned} 1 &> \tilde{x}(\tilde{\delta}(U)) \geq \tilde{x}(\tilde{\delta}(W \setminus X)) + \tilde{x}(\tilde{\delta}(X' \setminus W')) \geq \tilde{x}(\tilde{\delta}(W \setminus X)) \\ &= \sum_{v\in W\setminus X} \sum_{e\in\tilde{\delta}(v)} \tilde{x}(e) - \sum_{e\in E[W\setminus X]} 2\tilde{x}(e) = |W \setminus X| - \sum_{e\in E[W\setminus X} 2x(e) \end{aligned}$$

$$\Leftrightarrow \sum_{e\in E[W\setminus X} x(e) > \frac{|W \setminus X| - 1}{2}$$

Exercise 2

Let E be a finite set and let $\mathcal{I}$ be a non-empty collection of subsets of E such that $I \in \mathcal{I}$ and $J \subseteq I$ implies $J \in \mathcal{I}$. Prove that the following conditions are equivalent:

(i) if $I, J \in \mathcal{I}$ and $|J| > |I|$, then $I \cup \{e\} \in \mathcal{I}$ for some $e \in J \setminus I$;
(ii) if $I, J \in \mathcal{I}$ and $|J| = |I| + 1$, then $I \cup \{e\} \in \mathcal{I}$ for some $e \in J \setminus I$;
(iii) if $I, J \in \mathcal{I}$ and $|I \setminus J| = 1$, $|J \setminus I| = 2$, then $I \cup \{e\} \in \mathcal{I}$ for some $e \in J \setminus I$.
(iv) for all $A \subseteq E$, every maximal subset $I \subseteq A$ with $I \in \mathcal{I}$ has the same cardinality.

Solution

(i) $\Leftrightarrow$ (ii): Clear from definition.
(i) $\Leftrightarrow$ (iii): See Schrijver, Theorem 39.1 (part B, page 652)
(i) $\Leftrightarrow$ (iv): Seen in lecture.

Exercise 3 ($\star$)

Let $G = (V, E)$ be a graph. Let $\mathcal{I} \subseteq 2^V$ be defined as follows:
For $U \subseteq V$, we have $U \in \mathcal{I}$ if and only if there exists a matching in G that covers U (and possibly other vertices).
Show that $M = (V, \mathcal{I})$ is a matroid.

Solution

See Proposition 8.1 on page 276 of Cook et al. : Combinatorial Optimization.

Exercise 4
Let E be a finite set that is partitioned into sets $E = E_1 \cup \ldots \cup E_r$ and define a system

$$\mathcal{I} := \{S \subset E \mid |S \cap E_j| \leq 1 \text{ for all } j = 1 \ldots r\}.$$

of independent sets. Show that $(E, \mathcal{I})$ is a matroid. What is the rank of this matroid? Give a simple description of the bases of the matroid.
Remark: This type of matroid is called a *partition matroid*.

Solution

See Cook et al., page 284 and 285.

Exercise 5
Let $G = (V, E)$ be a graph and consider the Maximum Cardinality Matching problem.

(i) Show that the set system $(E, \mathcal{I})$ with $\mathcal{I} = \{M \subseteq E : M \text{ is a matching}\}$ is not a matroid. Hence, the Greedy algorithm (with respect to unit edge weights) will not necessarily produce a maximum matching.

(ii) Show that the Greedy algorithm applied to the above set system produces a solution that is at least half the size of an optimal solution.

Solution

(i) Consider $E := \{a, b, c\}$ and $\mathcal{I} := \{\{a, c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$ (This corresponds to a graph where the edges a,b,c form a path a,b,c). $(E, \mathcal{I})$ is not a matroid, since the independent set $\{b\}$ cannot be augmented by an element from the independent set $\{a, c\}$. Thus if Greedy picks edge b first, it would end up in a suboptimal solution.

(ii) Note that every maximum matching is also a maximal matching. Consider a maximal matching M that is not maximum. Observe that every edge of M shares a common node with at most two edges from any maximum matching. Notice further that every edge of a maximum matching shares at least one endpoint with an edge from M (by maximality of M). The Greedy algorithm finds a maximal matching, but by the previous arguments, its cardinality is at least half of the cardinality of a maximum matching.

Remark: It is also possible to show that in the weighted case, the Greedy algorithm always finds a matching of at least half of the optimum weight. However, this result needs a more involved analysis of the Greedy algorithm.