Prof. F. Eisenbrand EPFL - DISOPT

Combinatorial Optimization

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Sheet 2

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General remark:
In order to obtain a bonus for the final grading, you may hand in written solutions to the exercise marked with a star at the beginning of the exercise session on October 18.

Exercise 1

Let P be a rational polyhedron in $\mathbb{R}^n$. Show that if P equals the convex hull of its integral points, i.e. $P = \text{conv}(P \cap \mathbb{Z}^n)$, then P is integral.

Solution

Let F be a face of P, i.e. it exists a hyperplane H such that $F = P \cap H$. Consider a point $x \in F$. If P is the convex hull of its integral points, x is a convex combination of integral points in P. Those points must belong to H and thus to F.

Exercise 2

Consider the two following systems:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The two systems clearly define the same polyhedron. Show that the first one is TDI, but the second is not.

Solution

Consider objective $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ that can not be represented as an integer conic combination of the rows of the second system, the unique dual solution is $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$.
For the TDI-ness of the first system: the three lines of the matrix generate all integer points in the cone spanned by the first and the third line with integer coefficients. Note that this is not the case for the second system.

Exercise 3
Show the following:
If $Ax \le b$ is TDI and $ax \le \beta$ is a valid inequality for $\{x\colon Ax \le b\}$, then the system $Ax \le b, ax \le \beta$ is also TDI.

Hint: Use duality!

Solution

Let c be an integral vector such that the dual of the second system $\min\{yb + \gamma\beta\colon yA + \gamma a = c, y \ge 0, \gamma \ge 0\}$ is finite. Since $ax \le \beta$ is valid for $\{x\colon Ax \le b\}$,

$$\begin{aligned}\min\{yb\colon yA = c, y \ge 0\} &= \max\{cx\colon Ax \le b\}\\ &= \max\{cx\colon Ax \le b, ax \le \beta\}\\ &= \min\{yb + \gamma\beta\colon yA + \gamma a = c, y \ge 0, \gamma \ge 0\}.\end{aligned}$$

The first minimum is attained by some integral vector y^*, so $y = y^*$, $\gamma = 0$ is an integral optimum solution for the second minimum.

Exercise 4 ($\star$)

Consider a family $S_1, \dots, S_m$ of subsets $\{1, \dots, n\}$. The *set-covering problem* is to choose a smallest number of these sets whose union is $\{1, \dots, n\}$. This is modeled in the following integer linear program

$$\begin{aligned}&\min \textstyle\sum_{j=1}^m x_j\\ &A\,x \ge \mathbf{1}\\ &x \ge 0, \text{ integral}\end{aligned}$$

where $A \in \{0,1\}^{n\times m}$ is the matrix $A(i,j) = 1$ if $i \in S_j$ and $A(i,j) = 0$ otherwise.

The goal of this exercise is to see that things are not as nice here as in the case of maximum weight matchings. We cannot expect to prove optimality of an integral solution by providing an optimal dual solution.

(i) Provide an example where the linear program (integrality-constrained ignored) has a strictly smaller solution than the integer program.
(ii) Let x^* be an optimal solution to the linear program. We are now selecting some sets in $S_1, \dots, S_m$ at random: If $x_j^* \ge 1$, then select set S_j. Otherwise select S_j with probability x_j^*. What is the expected number of selected sets?
(iii) Show that the probability that a particular element $i \in \{1, \dots, n\}$ is not covered is bounded by $(1 - 1/m)^m \le e^{-1}$.
(iv) If this complete rounding procedure is repeated k-times, then the probability that i is not covered in any round is bounded by e^{-k}.
(v) If OPT denotes the optimum value of the integer linear program and OPT_f denotes the optimum value of the linear program, then conclude that $OPT/OPT_f = O(\log n)$.
(vi) Can you find a class of set-covering problems with $OPT/OPT_f = \Omega(\log n)$?

Solution

(i) Consider the universe $U := \{1, 2, 3\}$ and the set system $\mathcal{S} := \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$, i.e. $n = m = 3$. Choosing each set with value $1/2$ results in a feasible fractional solution of value 1.5 while every feasible integer solution must contain at least 2 sets.

(ii) Note first that the case $x_j^* > 1$ for some $j \in \{1, \dots, m\}$ can be excluded. We obtain

$$\mathbb{E}[\text{number of chosen sets}] = \sum_{j=1}^{m} 1 \cdot Pr[\text{set } S_j \text{ is chosen}] = \sum_{j=1}^{m} x_j^* = OPT_f$$

(iii) We have for a particular element i that

$$Pr[i \text{ is not covered}] = Pr[S_j \text{ is not chosen } \forall j : i \in S_j] = \Pi_{j:\, i \in S_j}(1 - x_j^*)$$
$$= \Pi_{j=1}^{m}(1 - A_{ij}x_j^*) = \Pi_{j=1}^{m}(1 - A_{ij}x_j^*)^{m/m} = \left(\sqrt[m]{\Pi_{j=1}^{m}(1 - A_{ij}x_j^*)}\right)^m$$
$$\leq \left(\frac{\sum_{j=1}^{m}(1 - A_{ij}x_j^*)}{m}\right)^m \leq \left(1 - \frac{1}{m}\right)^m \leq \frac{1}{e}$$

Note that the geometric mean is at most the arithmetic mean.

(iv) We repeat the process k times independently, thus

$$Pr[i \text{ is not covered in k rounds}] = (Pr[i \text{ is not covered}])^k \leq \left(\frac{1}{e}\right)^k = e^{-k}$$

(v) We choose $k = 2 \log n$ and we obtain

$$Pr[i \text{ is not covered in k rounds}] \leq \frac{1}{n^2} \leq \frac{1}{4n}$$

for $n \geq 4$. This implies by the union bound:

$$Pr[\text{ Picked sets after k rounds is not a feasible cover}]$$
$$\leq \sum_{i=1}^{n} Pr[i \text{ is not covered in k rounds}] \leq \frac{1}{4}$$

The expected number of picked sets obviously is

$$k \cdot \mathbb{E}[\text{number of chosen sets per round}] = 2 \log n OPT_f.$$

Furthermore, the probability that the number of sets exceeds $O(\log n)OPT_f$ is small (using Markov's inequality):

$$Pr[\text{number of sets picked} \geq 4\mathbb{E}[\text{number of sets picked}]]$$
$$\leq \frac{\mathbb{E}[\text{number of sets picked}]}{4\mathbb{E}[\text{number of sets picked}]} = \frac{1}{4}$$

Together, we have that we pick $O(\log n)OPT_f$ sets that are a feasible cover with probability $\geq 1/2$. Thus there is an integer solution that verifies this characterization (by the so called probabilistic method) and we obtain the upper bound of $O(\log n)$ on the integrality, i.e. $\frac{OPT}{OPT_f}$.

Remark: We can turn the randomized algorithm described in the previous points into an approximation algorithm for the Set Cover problem with approximation factor $O(\log n)$, since the expected number of repetitions of the randomized algorithm to obtain a good cover is 2.

(vi) We will use the linear algebra method to construct an example set cover instance with a fractional optimum very close to 2, and an integer optimum close to $\log n$, where n is the number of points.

As set of points we are going to use the set of non-zero vectors of a k-dimensional vector space over the field $\mathbb{F}_2$:

$$V := \mathbb{F}_2^k \setminus \{0\}$$

The sets will also be indexed by non-zero vectors from $\mathbb{F}_2^k$. For every $u \in \mathbb{F}_2^k \setminus \{0\}$, we define a set

$$S_u := \{v \in V \mid u^T v = 1\}$$

Note that the scalar product $u^T v$ is computed in $\mathbb{F}_2$, so the computation is done modulo 2. The points in S_u form a $(k-1)$-dimensional affine subspace, and so

$$|S_u| = 2^{k-1}$$

Since the construction has a nice symmetry,[1] every point is contained in exactly 2^{k-1} sets. In other words, we obtain a feasible fractional solution by setting all variables to the same value:

$$x_u = \frac{1}{2^{k-1}}$$

The objective function value of this solution is

$$\sum_{u \in \mathbb{F}_2^k \setminus \{0\}} x_u = \frac{2^k - 1}{2^{k-1}} = 2 - \frac{1}{2^{k-1}}$$

Let us now prove that the optimal integer solution uses exactly k sets. Let $e_1, \ldots, e_k \in \mathbb{F}_2^k$ be the unit vectors, then the sets $S_{e_1}, \ldots, S_{e_k}$ cover all points of V. Simply observe that every $v \in V$ has at least one non-zero entry, say in the j-th coordinate. Then $e_j^T v = 1$, and therefore $v \in S_{e_j}$.

Now suppose we have sets $S_{u_1}, \ldots, S_{u_r}$ with $r < k$. We claim that there is at least one point in V that is not covered by these sets. To see this, consider the system of linear equations:

$$\begin{aligned} u_1^T x &= 0 \\ &\vdots \\ u_r^T x &= 0 \end{aligned}$$

Since $r < k$, the set of solutions is a subspace of dimension at least 1. Therefore, there exists a non-zero solution $v \in V$. Furthermore, $u_j^T v = 0$ for all j implies $v \notin S_{u_j}$ for all j, which means that v is not covered. Keeping in mind that $n = |V| = 2^k - 1$, we conclude that

$$\frac{OPT}{OPT_f} = \frac{k}{2 - \frac{1}{2^{k-1}}} \geq \frac{k}{2} = \Omega(\log n)$$

[1] This can be stated formally: if we use the same ordering of vectors for both points and sets, then the coefficient matrix A of the linear program is symmetric. This is just a side effect of the fact that the scalar product is symmetric.

Remark: If you think about the above system of linear equations a little, you will soon realize that *every* basis of $\mathbb{F}_2^k$ corresponds to an optimal solution of the set cover instance. So the number of optimal solutions is huge, and they are very evenly distributed. This is not an accident: the linear algebra method is typically used for problems in combinatorics where the number of "optimal" or "extremal" configurations is very large.