Prof. F. Eisenbrand EPFL - DISOPT

# Combinatorial Optimization

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## Sheet 1 September 22, 2011

General remark:
In order to obtain a bonus for the final grading, you may hand in written solutions to the exercise marked with a star at the beginning of the exercise session on October 4.

### Exercise 1

Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Show that the dual of the linear program

$$\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \le b,\ x \ge 0\}$$

can be interpreted as the linear program

$$\min\{b^T y \colon y \in \mathbb{R}^m,\ A^T y \ge c,\ y \ge 0\}.$$

In addition, mark (and argue!) the entries of the following table corresponding to possible outcomes in this primal/dual pair of linear programs:

| | | Dual | | |
|---|---|---|---|---|
| | | Finite optimum | Unbounded | Infeasible |
| Primal | Finite optimum | | | |
| | Unbounded | | | |
| | Infeasible | | | |

### Solution

First part: Understand the primal as $\max\{c^T x \colon x \in \mathbb{R}^n, \left(\begin{smallmatrix} A \\ -I \end{smallmatrix}\right) x \le \left(\begin{smallmatrix} b \\ 0 \end{smallmatrix}\right)\}$, and re-interpret the dual of this LP.
Second part:

- Primal and dual feasible and bounded is *possible*: Example is $c = b = (0)$ and $A = (0)$.
- Primal feasible and bounded, dual unbounded is *impossible*: Assume that $Ax \le b$ is has a solution $x$. Then by weak duality, $c^T x$ a lower bound for all solutions to the dual, in contradiction to the fact that the dual is unbounded.
- Primal feasible and bounded, dual infeasible is *impossible*: If the primal has an optimal solution, the duality theorem tells us that the dual has an optimal solution as well. In particular the dual is feasible.
- Primal unbounded and dual feasible and bounded is *impossible*: Assume that $A^T y = c$ has a solution $y$. Then by weak duality, $b^T y$ is an upper

bound for all solutions to the primal, in contradiction to the fact that the primal is unbounded.

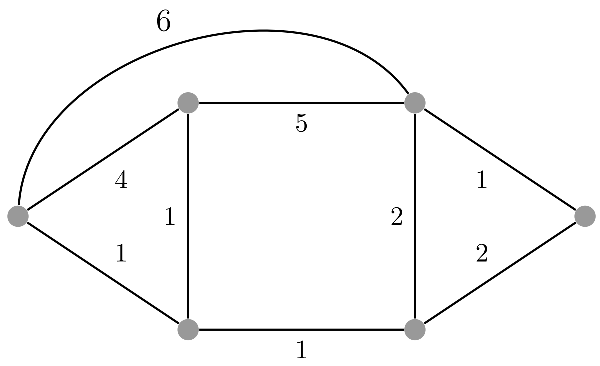
- Primal unbounded, dual unbounded is *impossible*: As seen above, if the primal is unbounded, then the dual is infeasible.
- Primal unbounded, dual infeasible is *possible*: Example is $c = (1)$, $b = (0)$ and $A = (0)$.
- Primal infeasible, dual feasible and bounded is *impossible*: With the strong duality theorem, if the dual is feasible and bounded, so is the primal.
- Primal infeasible, dual unbounded is *possible*: Example is $c = (0)$, $b = (-1)$ and $A = (0)$.
- Primal and dual infeasible is *possible*: Example is $c = (1)$, $b = (-1)$ and $A = (0)$.

**Exercise 2**

Determine a maximum weight matching of the graph below. Provide of proof of optimality by determining a feasible dual solution to the linear program

$$\begin{array}{rrcl} \max & \sum_{e \in E} w(e)x(e) & & \\ v \in V: & \sum_{e \in \delta(v)} x(e) & \leq & 1 \\ {\scriptstyle U \subseteq V \atop |U| \text{ odd}}: & \sum_{e \in E(U)} x(e) & \leq & \lfloor |U|/2 \rfloor \\ e \in E: & x(e) & \geq & 0 \end{array}$$

whose objective value coincides with the weight of your matching.

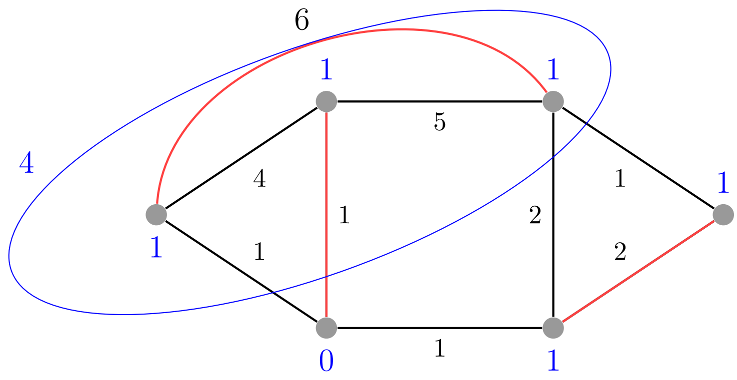



**Solution**

The dual is

$$\begin{array}{rrcl} \min & \sum_{v \in V} y(v) + \sum_{U \subseteq V \atop |U| \text{ odd}} \lfloor |U|/2 \rfloor z(U) & & \\ e \in E: & \sum_{v \in e} y(v) + \sum_{U \supseteq e \atop |U| \text{ odd}} z(U) & \geq & w(e) \\ {\scriptstyle U \subseteq V \atop |U| \text{ odd}}: & z(U) & \geq & 0 \\ v \in V: & y(v) & \geq & 0 \end{array}$$

A feasible primal solution picks the red edges and has value 9. A feasible dual solution is denoted in blue and has value 9. Thus both solutions are optimal.




**Exercise 3** (⋆)

In the following, we consider undirected graphs without edge weights.

Suppose that we have an efficient algorithm to decide whether a graph contains a perfect matching (i.e. each node is incident with exactly one edge).
Use that algorithm to determine efficiently the size of a maximum matching in a given graph $G = (V, E)$.

**Solution**
Let $\mathcal{A}$ be the efficient algorithm to decide whether a given graph contains perfect matching. We will present an algorithm for determining the size of a maximum matching in $G$ that uses $O(|V|$ calls to $\mathcal{A}$.

MAXIMUM-MATCHING-SIZE$(G = (V, E))$

1 **if** $\mathcal{A}(G) = $ TRUE
2     **return** $\frac{|V|}{2}$
3 ▷ We assume that $V = \{v_1, \ldots, v_{|V|}\}$.
4 $G' = G$
5 **for** *step* $\leftarrow 1$ **to** $|V|$
6     $V' = V' \cup x_{step}$
7     $E' = E' \cup \{\{v_i, x_{step}\} : v_i \in V\}$
8     $G' = (V', E')$
9     **if** $\mathcal{A}(G') == $ TRUE
10         **return** $\frac{|V| - step}{2}$

In order to prove correctness of the algorithm we have to show the following:

a) The algorithm terminates, and
b) value returned by the algorithm is the size of a maximum matching of $G$.

**a)** Let us consider the last iteration, i.e. the graph augmented by $|V|$ artificial nodes. By the constructions in the lines 6 and 7, $G'$ contains edges $\{v_i, x_i\}, i = 1..|V|$ and thus a perfect matching.

**b)** Assume that the algorithm stops with $step = t$ (The correctness is trivial if the algorithms stops in line 2). We show that the following holds:

i) Matching of size $\frac{|V|-t}{2}$ exists in $G$, and

ii) matching of the size $\frac{|V|-t}{2}$ is the maximum in $G$.

**i)** In a perfect matching of $G'$, each vertex $x_k, k = 1..t$, must be joined to a vertex $v_{i_k} \in V$, since by the construction, there is no edge between $x_i$ and $x_j$, $1 \leq i, j \leq t$. The remaining $|V| - t$ vertices, all in $V$, are covered by a matching.

**ii)** Let us assume that there exists a matching of size $S > \frac{|V|-t}{2}$. Assume wlog that the vertices $v_1, \dots, v_{2S}$ form that matching. Let $t' = |V| - 2S$. Clearly $t' < t$. For $step = t'$ all the vertices $v_{2S+1}, \dots, v_{2S+t'}$ are matched to the vertices $x_1, \dots, x_{t'}$, thus $v_{2S+1}, \dots, v_{2S+t'}, x_1, \dots, x_{t'}$ are perfectly covered by a matching of size $t'$. It implies that $G'$ for $step = t'$ contains a perfect matching, a contradiction.

Note that it is possible to reduce the number of calls to $\mathcal{A}$ in the worst case to $\frac{|V|}{2}$, since you only have to consider the cases in which the total number of nodes is even (otherwise there trivially is no perfect matching).

### Exercise 4

Show the following: A face $F$ of $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ is inclusion-wise minimal if and only if it is of the form $F = \{x \in \mathbb{R}^n \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.

### Solution

Let $F$ be an inclusion-wise minimal face. Write

$$F = \{x \in P \colon A'x = b'\}$$

where $A'x \leq b'$ is the maximal possible subsystem of $Ax \leq b$ with that property, and let

$$G = \{x \in \mathbb{R}^n \colon A'x = b'\}$$

Assume $F \neq G$, then there is a point $z \in G \setminus F$. In particular, $z \notin P$. Furthermore, there exists a point $y \in F$. Consider the line segment parameterized by:

$$w(t) = (1-t)y + tz, t \in [0, 1]$$

Let $a^T x \leq \beta$ be the first inequality of $Ax \leq b$ that is violated as $w(t)$ moves from $y$ to $z$, and let $t \in [0, 1)$ such that $a^T w(t) = \beta$. Then

$$F' = \{x \in P \colon A'x = b', a^T x = \beta\}$$

is a face of $P$ by the theorem seen in lecture 1, it is clearly contained in $F$, and it is non-empty because $w(t) \in F'$. Finally, note that $a^T x = \beta$ cannot be contained in the system $A'x = b'$, because $a^T w(t) = \beta$ does not hold for all $t \in [0, 1]$. Therefore, $F'$ is defined by a subsystem of equations that is strictly bigger than any subsystem that defines $F$ (remember that we chose $A'x = b'$ to be maximal!) and so $F' \neq F$. In conclusion, $F'$ is a proper sub-face of $F$, which contradicts the inclusion-wise minimality of $F$. So the assumption was wrong, in fact we have $F = G$.

Let $F$ be a face of $P$ such that $F = \{x \in \mathbb{R}^n \colon A'x = b'\}$ for a subsystem $A'x \leq b'$ of $Ax \leq b$. Assume that $F$ is not inclusion-wise minimal, i.e. there is a proper sub-face $F' \subseteq F$. We can write

$$F' = \{x \in P \colon A'x = b', A''x = b''\}$$

for a subsystem $A''x \leq b''$ of $Ax \leq b$. Let $y \in F'$ and $z \in F \setminus F'$. Then the line through $y$ and $z$ is contained entirely in $F$, however there will be one inequality $a^T x \leq \beta$ of the system $A''x \leq b''$ that is not parallel to the line through $y$ and $z$. This means that the line cannot be entirely contained in $P$. This is a contradiction, and so $F$ must be inclusion-wise minimal.