Goals

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- **Recap:** $s - t$-flows, $s - t$-cuts, weak duality
- Ford-Fulkerson algorithm
- Strong duality: Max-Flow-Min-Cut-Theorem
- Edmonds-Karp algorithm
- Application: Scheduling on Uniform Parallel Machines

Recap: $s - t$-flows

- Network: Digraph $D = (V, A)$ with capacity function $u: A \to \mathbb{R}_{\geq 0}$.
- For $s, t \in V$, an $s - t$-flow in D is a function $f: A \to \mathbb{R}_{\geq 0}$ such that

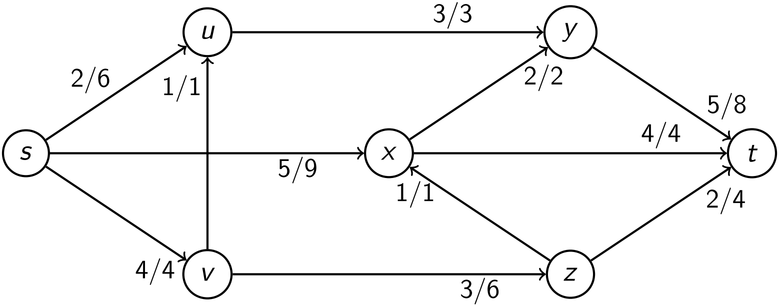
$$\sum_{a \in \delta^{out}(v)} f(a) = \sum_{a \in \delta^{in}(v)} f(a), \text{ for all } v \in V - \{s, t\}.$$

- f is feasible , if $f(a) \leq u(e)$ for all $a \in A$.
- The value of f is

$$\text{value}(f) := \sum_{a \in \delta^{out}(s)} f(a) - \sum_{a \in \delta^{in}(s)} f(a).$$

Recap: $s - t$-flows

Example:



Maximum $s - t$-flow problem

Find a feasible $s - t$-flow of maximum value.

Recap: $s-t$-cuts

$D=(V,A)$, $u:\ A\to\mathbb{R}_{\geq 0}$

- $U\subseteq V$: $\delta^{out}(U):=\{(u,v)\in A\ :\ u\in U,\ v\notin U\}$ is a cut .
- If $s\in U$, $t\notin U$: $\delta^{out}(U)$ is an $s-t$-cut .
- Capacity : $u(\delta^{out}(U)):=\sum_{a\in\delta^{out}(U)}u(a)$.

Minimum $s-t$-cut problem

Find an $s-t$ cut $\delta^{out}(U)$ such that $u(\delta^{out}(U))$ is minimal.

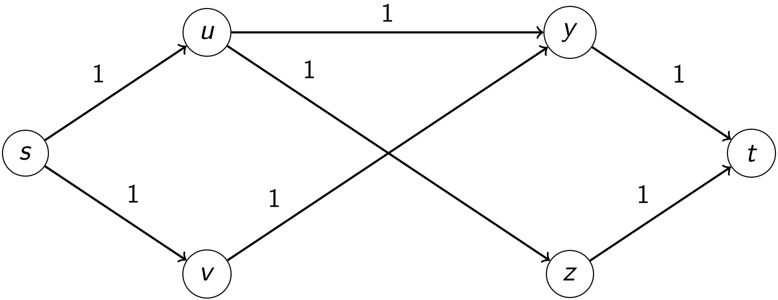
Theorem (Weak duality)

Let f be a feasible $s-t$-flow and let $\delta^{out}(U)$ be an $s-t$-cut, then

$$\text{value}(f)\leq u(\delta^{out}(U)).$$

Computing max flows: Ford-Fulkerson algorithm

Informal idea: While there is an $s - t$-path with "positive capacity" in the graph, send as much flow as possible along the path.



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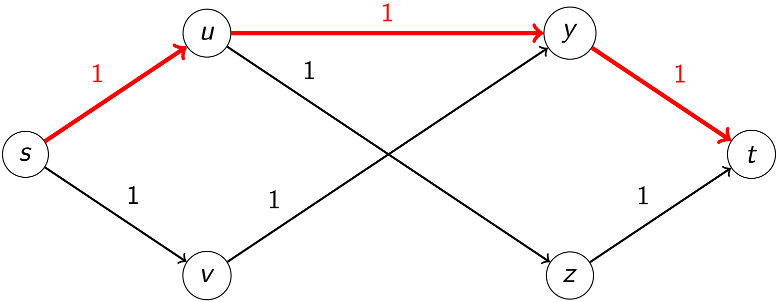

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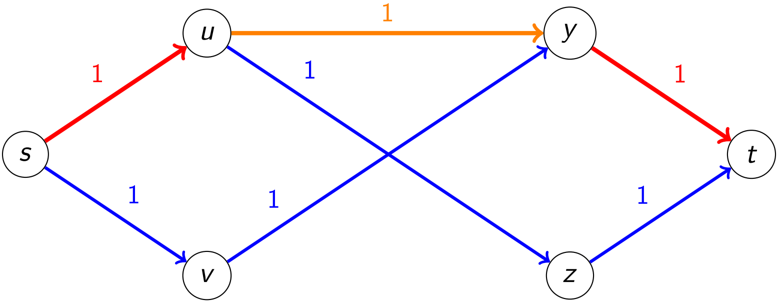

Computing max flows: Ford-Fulkerson algorithm

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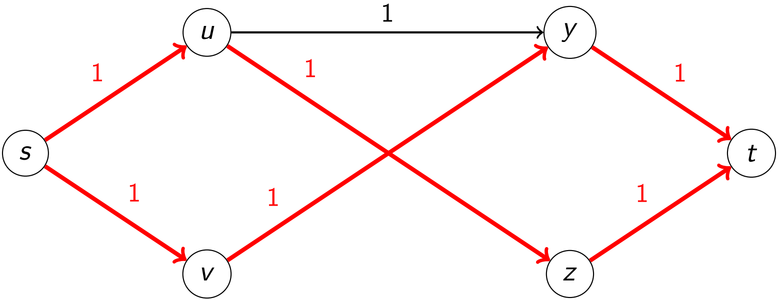


Residual graphs

- For an arc $a = (u, v) \in A$, let a^{-1} denote the arc (v, u).
- **Wlog.**: $a^{-1} \notin A$.
- Let $f : A \to \mathbb{R}$ and $u : A \to \mathbb{R}_{\geq 0}$ with $0 \leq f \leq u$.
- Define

$$A_f := \{a \ : \ a \in A, f(a) < u(a)\} \cup \{a^{-1} \ : \ a \in A, f(a) > 0\}.$$

- $D_f = (V, A_f)$ is the **residual graph** of f.
- Define **residual capacities**

$$u_f : A_f \to \mathbb{R}_{\geq 0}, \ u_f(a) = \begin{cases} u(a) - f(a), & \text{if } a \in A \\ f(a), & \text{if } a^{-1} \in A. \end{cases}$$

Ford-Fulkerson algorithm

- An **undirected path** is a sequence $P = (v_0, a_1, v_1, \ldots, v_{m-1}, a_m, v_m)$ such that $a_i \in A$ and $a_i = (v_{i-1}, v_i)$ or $a_i = (v_i, v_{i-1})$ for each $i = 1, \ldots, m$.
- Every directed path P in D_f yields an undirected path in D.
- Define $\chi^P \in \{0, \pm 1\}^A$ as

$$\chi^P(a) = \begin{cases} 1, & \text{if } P \text{ traverses } a, \\ -1, & \text{if } P \text{ traverses } a^{-1}, \\ 0, & \text{if } P \text{ traverses neither } a \text{ nor } a^{-1}. \end{cases}$$

function $\textsc{Ford-Fulkerson}(D = (V, A),\ u)$
    $f \leftarrow 0$.
    while $\exists\ s - t$-path P in D_f **do**
        $\varepsilon \leftarrow \min\{u_f(a)\ :\ a \in P\}$.
        $f \leftarrow f + \varepsilon \cdot \chi^P$.
    end while
    return f.
end function

Ford-Fulkerson: Correctness

function FORD-FULKERSON$(D = (V, A),\ u)$
  $f \leftarrow 0$.
  while $\exists\ s - t$-path P in D_f **do**
    $\varepsilon \leftarrow \min\{u_f(a)\ :\ a \in P\}$.
    $f \leftarrow f + \varepsilon \cdot \chi^P$.
  end while
  return f.
end function

Theorem

The output f of FF is a flow.

Theorem

If u is integer, then the output f of FF is integer.

Strong duality

Recall: $D = (V, A)$, $u : A \to \mathbb{R}_{\geq 0}$, f flow, $U \subseteq V$

- $\text{excess}_f(U) := \sum_{a \in \delta^{in}(U)} f(a) - \sum_{a \in \delta^{out}(U)} f(a)$.
- $\text{excess}_f(U) = \sum_{v \in U} \text{excess}_f(v)$.
- $A_f := \{a : a \in A, f(a) < u(a)\} \cup \{a^{-1} : a \in A, f(a) > 0\}$.

Theorem (Max-Flow-Min-Cut-Theorem)

The maximum value of a feasible $s-t$-flow is equal to the minimum capacity of an $s-t$-cut.

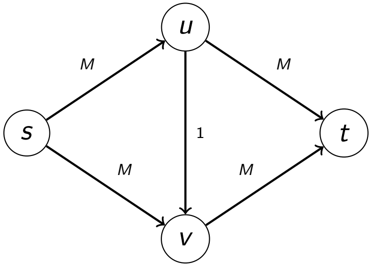
Corollary

FF computes a maximum $s-t$-flow. If u is integer, there is an integer max flow.

Running time

```
function Ford-Fulkerson(D = (V, A), u)
    f ← 0.
    while ∃ s − t-path P in D_f do
        ε ← min{u_f(a) : a ∈ P}.
        f ← f + ε · χ^P.
    end while
    return f.
end function
```

- Each iteration takes time $O(|A|)$ (using e.g. Breadth-First search).
- How many iterations?



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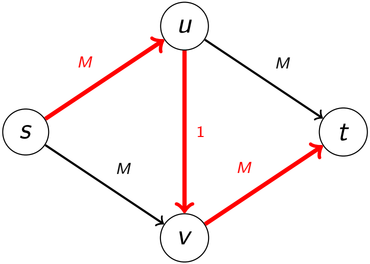

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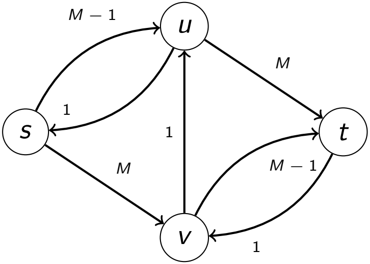

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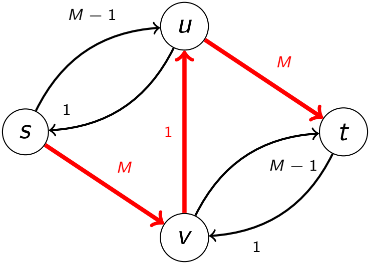

Running time

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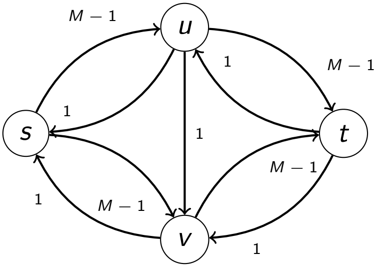
function FORD-FULKERSON(D = (V, A), u)
    f ← 0.
    while ∃ s − t-path P in D_f do
        ε ← min{u_f(a) : a ∈ P}.
        f ← f + ε · χ^P.
    end while
    return f.
end function
```

- Each iteration takes time $O(|A|)$ (using e.g. Breadth-First search).
- How many iterations?



Emonds-Karp: Max-Flow in polynomial time

Theorem

If we choose in each iteration a shortest $s-t$-path in D_f as a flow augmenting path, the number of iterations is at most $|V| \cdot |A|$.

- Digraph $D = (V, A)$, $s, t \in V$
- $\mu(D)$ is length of a shortest path from s to t.
- $\alpha(D)$ is the set of arcs contained in at least one shortest $s-t$-path.

Lemma

Let $D = (V, A)$ be a digraph and $s, t \in V$. Define $D' := (V, A \cup \alpha(D)^{-1})$. Then $\mu(D) = \mu(D')$ and $\alpha(D) = \alpha(D')$.

Corollary

A maximum $s-t$-flow an be found in time $O(|V| \cdot |A|^2)$.

Scheduling on Uniform Parallel Machines

The setting

- n jobs, job j characterized by
  - processing time p_j,
  - release time r_j,
  - deadline d_j.
- M identical machines.
- Each machine can process only one job at a time.
- Each job requires a total processing time p_j between release time r_j and deadline d_j.
- Preemption allowed, i.e. processing of a job can be interrupted and continued later.

Scheduling problem

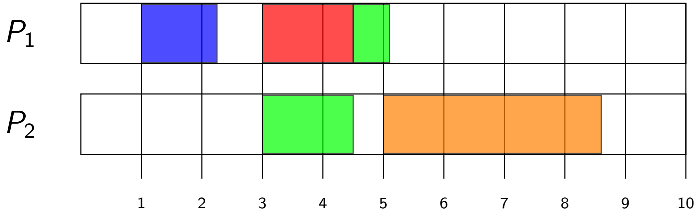
Find a feasible schedule that meets all constraints or assert that no such schedule exists.

Scheduling on Uniform Parallel Machines

Example:

Job (j)	1	2	3	4
p_j	1.5	1.25	2.1	3.6
r_j	3	1	3	5
d_j	5	4	7	9

Feasible schedule on 2 processors:



What does this have to do with Max-Flows?

Scheduling on Uniform Parallel Machines

- Sort the release times and deadlines ascending. Let $t_1, \ldots, t_k$ be the sorted sequence. (In our example: 1 3 4 5 7 9)
- Split the time horizon into disjoint intervals, i.e. define intervals $T_i := [t_i, t_{i+1})$ for $i = 1, \ldots, k$.
  (In our example we have $T_1 = [1, 3)$, $T_2 = [3, 4)$, $T_3 = [4, 5)$, $T_4 = [5, 7)$, $T_6 = [7, 9)$).
- Construct a network as follows:
  - Insert a node j for each job $j = 1, \ldots, n$.
  - Insert a node T_i for each interval T_i, $i = 1, \ldots, k - 1$.
  - Add an arc (j, T_i) for each job j such that $T_i \subseteq [r_i, d_i]$ of capacity $|T_i|$.
  - Add a node s and arcs (s, j) for each job j, of capacity p_j.
  - Add a node t and arcs (T_i, t) for each interval i, of capacity $M \cdot |T_i|$.

Theorem

There is a feasible schedule for the scheduling problem if and only if there is a flow of value $\sum_{j=1}^{n} p_j$.

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- Strong duality: Max-Flow-Min-Cut-Theorem ✓
- Edmonds-Karp algorithm ✓
- Application: Scheduling on Uniform Parallel Machines ✓