Optimization Methods in Finance	(EPFL, Fall 2010)

Lecture 12: Two-person Games, Brouwer's Fix Point Theorem

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Introduction

In this lecture notes, we are going to present the two-person general sum games, illustrated by the Prisoner's Dilemma example. Looking at game theory, we then make a link between a game involving two or more players and a special solution concept named Nash equilibrium. We finish by the Brouwer's fixed point theorem : we prove it elementary using the Sperner's Lemma.

Two person general sum games

Two-person games, both zero-sum and (general) non-zero-sum, have been subjected to extensive analysis from a formal point of view. On the other hand, very little is known about how people actually behave in game situation, even those of a very simple type.
We can link this part to some examples related to economic theory: the struggle between labor and management, the competition between two producers of a single good, the negotiations between buyer and seller, and so on.

Let's set up the following game with an example at last: we consider two players:

- Player I: defined as a row player,
- Player II: defined as a column player.

We can now describe a finite two person game by a pair of payoff matrices A and B.
Indeed, if m and n representing the number of pure strategies of the two players, the game may be represented by two $m \times n$ matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times n}$. The interpretation here is that if Player I chooses row i and Player II chooses column j, then Player I wins a_{ij} and Player II wins b_{ij}, where a_{ij} and b_{ij} are the elements in the i^{th} row, j^{th} column of A and B respectively.

Thus, if $X \in \mathbb{R}^m_{\geq 0}$ and $Y \in \mathbb{R}^n_{\geq 0}$ are mixed strategies for row and column player respectively, then

- $X^T AY$ is the expected payoff for the row player,
- $X^T BY$ is the expected payoff for the column player.

To illustrate this formal set up, let's look at the example below:

Example 1. Prisoner's Dilemma

The prisoner's dilemma is a fundamental problem in game theory that demonstrates why two people might not cooperate even if it is in both their best interests to do so. It was originally framed by Merrill Flood and Melvin Dresher in 1950. Albert W. Tucker formalized the game with prison sentence payoffs and gave it the "prisoner's dilemma" name (Poundstone, 1992).

	Prisoner II confesses	Prisoner II stays silent
Prisoner I confesses	Each serves 4 years	Prisoner II: 5 year Prisoner I: 1 year
Prisoner I stays silent	Prisoner II: 1 year Prisoner I: 5 year	Each serves 2 years

Table 1 : Example of a classical prisoner's dilemma

According to Table 1, the prisoner's dilemma can be presented as follows:

Two suspects are arrested by the police. The police has insufficient evidence for a conviction, and, having separated the prisoners, visit each of them to offer the same deal.

- If one confess and testifies for the prosecution against the other (defects) and the other remains silent (cooperates), the defector receives a minimized 1-year sentence and the silent accomplice receives the full 5-year sentence.
- If both remain silent, both prisoners are sentenced to only 2 years in jail for a minor charge.
- If each betrays the other, each receives a 4-year sentence.

Each prisoner must choose to betray the other or to remain silent. Each one is assured that the other would not know about the betrayal before the end of the investigation. How should the prisoners act?

In this game, regardless of what the opponent chooses, each player always receives a higher payoff (lesser sentence) by betraying; that is to say that betraying is the strictly dominant strategy. Thus, the only stable pure strategy is : {Prisoner I confessess, Prisoner II confesses}.

For instance, Prisoner I can accurately say, "No matter what Prisoner II does, I personally am better off betraying than staying silent. Therefore, for my own sake, I should betray." However, if the other player acts similarly, then they both betray and both get a lower payoff than they would get by staying silent. Rational self-interested decisions result in each prisoner being worse off than if each chose to lessen the sentence of the accomplice at the cost of staying a little longer in jail himself (hence the seeming dilemma). In game theory, this demonstrates very elegantly that in a non-zero-sum game a Nash equilibrium need not be a Pareto optimum (we say that an outcome of a game is Pareto optimal if there is no other outcome that makes every player at least as well off and at least one player strictly better off).

Nash Equilibrium

In game theory, Nash equilibrium (named after John Forbes Nash, who proposed it) is a solution concept of a game involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his own strategy unilaterally. If each player has chosen a strategy and no player can benefit by changing his strategy while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitute a Nash equilibrium.

Formally, a Nash equilibrium is a pair of mixed strategies X,Y such that:

- whatever the mixed strategy X' : $X'^T AY \leq X^T AY$,
- whatever the mixed strategy Y' : $X^T BY' \leq X^T BY$.

In this part, our goal is to prove the existence of the Nash equilibrium.

Lemma 2. *The best response to the column player's mixed strategy is a pure strategy.*
Similarly, the best response to the row player's mixed strategy is a pure strategy.

Proof. Let $Y \in \mathbb{R}^n_{\geq 0}$ be a mixed column strategy. The expected payoff for the row player with strategy X is $X^T AY = X^T AY$. The best response is thus $X = (0,\dots,0,1,0,\dots,0)'$ (a 1 in the i^{th} position), with $i = \text{argmax}_{i=1,\dots,m}(AY)_i$ □

Lemma 3. *Given mixed strategies X and Y, there are always pure strategies X' and Y' such that:*

- $X'^T AY \leq X^T AY$
- $X^T BY' \leq X^T BY$

Proof. The proof is very similar to the previous one. (worst response is also a pure strategy)
To be noticed : X and Y are convex combinations of pure strategies (unit vectors). □

Theorem 4. ***Nash theorem :***
Every two-person general sum game has a Nash Equilibrium.

For this theorem, we need the following one:

Theorem 5. ***Brouwer's fix point theorem :***
if $S \subseteq \mathbb{R}^n$ is convex and compact, $f : S \to S$ continuous, then $\exists s \in S,\ f(s) = s$

Proof. Proof of the Nash's Theorem. □

Let X and Y be mixed strategies. We define

- $r_i(X,Y) = max\left\{0,\ e_i^T AY - X^T AY\right\}$,

- $c_i(X,Y) = max\left\{0,\ X^T B e_j - X^T BY\right\}$,

  where e_i and e_j are the unit vectors (i.e. 0 everywhere, except a 1 in i^{th} position and j^{th} position respectively).

To be noticed :
X,Y is a nash equilibrium$\Leftrightarrow\left\{\forall i = 1,\ldots,m,\ r_i(X,Y) = 0\ and\ \forall j = 1,\ldots,n,\ c_j(X,Y) = 0\right\}$
(in short $r = c = 0$).

We design now new strategies X' and Y' out of X and Y:

- $X'_i = \frac{X_i + r_i}{1+\sum_{k=1}^m r_k}$, where r_i is the abbreviation for $r_i(X,Y)$,
- $Y'_i = \frac{Y_j + c_j}{1+\sum_{k=1}^m c_k}$, where c_j is the abbreviation for $c_j(X,Y)$.

$X'$and Y' are mixed strategies since :

$$X' \geq 0$$
$$Y' \geq 0$$
$$\sum_{i=1}^m X'_i = \frac{\sum_{i=1}^m X_i + \sum_{i=1}^m r_i}{1+\sum_{k=1}^m r_k} = 1$$
$$\sum_{j=1}^m Y'_j = \frac{\sum_{j=1}^m X_i + \sum_{j=1}^m c_j}{1+\sum_{k=1}^m c_k} = 1$$

Interpretation : "Move" X and Y in improving direction, if not all the r_i and c_j are zero.

Consider the map: $T(X,Y) = (X',Y')$

We observe that T is a continuous map and $S = (X,Y) \in \mathbb{R}^{n+m} : X,Y$ mixed strategies is a convex and compact set.

We are done, once we show this claim: $(X',Y') = (X,Y) \Leftrightarrow (X,Y)$ is a nash equilibrium.

Suppose that (X,Y) is not a nash equilibrium. We thus show that: $(X',Y') \neq (X,Y)$.

If (X,Y) is not a nash equilibrium, then:

- $\exists i,\ e_i^T AY > X^T AY$ (*), or
- $\exists j,\ XBe_j > X^T BY$ (**).

Assume (*) holds,

- $X'_i = \frac{X_i + r_i}{1+\sum_{k=1}^m r_k}$, with $r_i \geq 0$

But there exists an index μ (worst pure response to Y) with $r_\mu = 0$ so that : $X'_\mu = \frac{X_\mu + 0}{1+\sum_{k=1}^m r_k} \neq X_\mu$.

(For (**) the argument is similar)

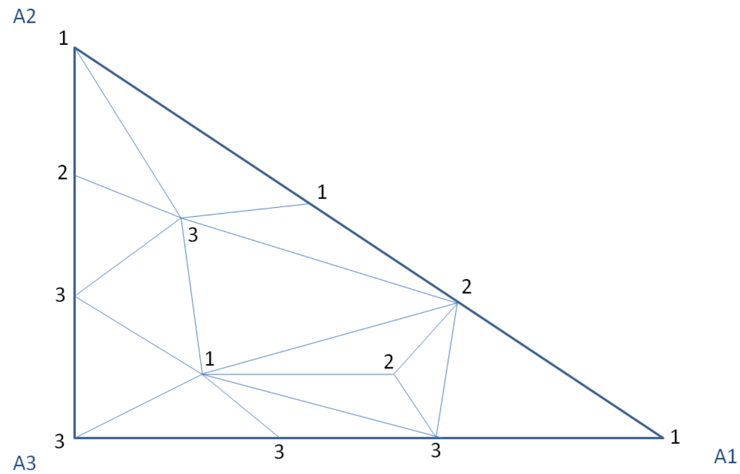

Figure 1

Why does Brouwer's fix point theorem hold?

Brouwer's fixed point theorem is a fixed point theorem in topology, named after Luitzen Brouwer. It states that for any continuous function with certain properties there is a point x_0 such that $f(x_0) = x_0$.

The simplest form of Brouwer's theorem is for continuous functions from a disk to itself.

A more general form is for continuous functions from a convex compact subset of Euclidean space to itself.

We provide an elementary proof for the case where $S \subseteq \mathbb{R}^2$ is the unit triangle.

The main tool is Sperner's Lemma: a triangulation of a triangle is a subdivision of the triangle into small triangles. The small triangles are called baby triangles, and the corners of the triangles are called vertices. The additional condition that each edge between two vertices is part of at most two triangles is necessary.

A Sperner Labeling is a labeling of a triangulation of a triangle with the numbers 1, 2 and 3 such that:

(1) The three corners are labeled 1, 2 and 3.
(2) Every vertex on the line connected vertex i and vertex j is labeled i or j.

Settings (see Figure 1):

(1) triangle with vertices A_1, A_2, A_3
(2) finite set of points in triangle with labels in $1, 2, 3$
(3) points in segment $\overline{A_i A_j}$ have labels i, j

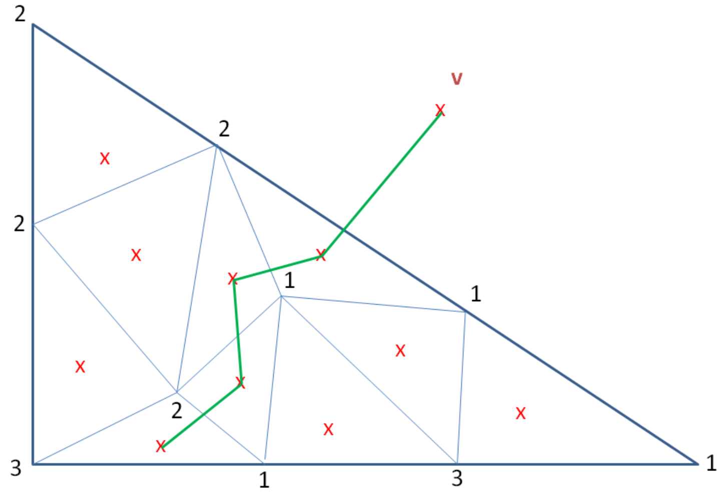

Figure 2

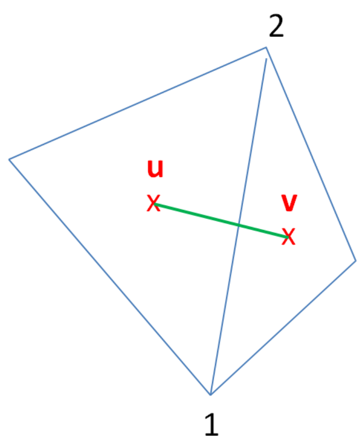

Figure 3

Theorem 6. *There exists a rainbow (baby) triangle (i.e. with corners labeled* 1, 2 *and* 3*) in any triangulation.*

Proof. We consider a sub-graph of the dual-graph provided by triangulation (each interior of a triangle hosts a vertex, and the "outside" hosts a vertex as well, we call it v) □

$u_v \in E \Leftrightarrow$The face of u and v share an edge of triangulation with end points labeled 1 and 2.

(For (**) the argument is similar)

Recall : $G = (V,E)$ Graph, then : $\sum_{v\in V}\ diag(v) = 2|E| \Rightarrow$ even

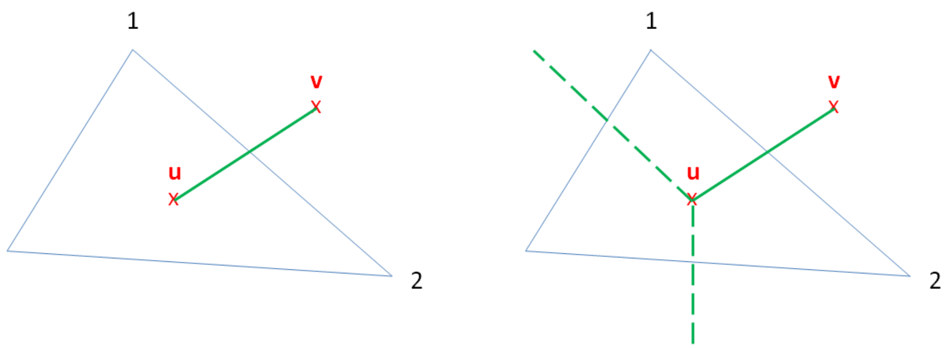

Figure 4

Since the degree of v is odd, there must exist another node in the interior of a triangle with odd degree. In fact, degree one $\Rightarrow$ rainbow triangle.

Comments (see Figure 4):

- on the triangle of the left : the corner that is not labelled must be 3
- on the triangle of the right : the line between u and v cannot be the two dashed line, otherwise the corner that is not labelled should be 1, and this is not possible.

References

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