Optimization Methods in Finance **(EPFL, Fall 2010)**

Lecture 7: Linear programming, Dedicated Bond Portfolios

03.11.2010

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Linear programming is an important tool with which one can model and solve many important problems in the world of finance. The next couple of lectures will be devoted to linear programming.

Linear Programming

A *linear program* is a convex optimization problem of the form

$$\begin{aligned} \min c^T x & \\ \text{s.t.} \quad a_i^T x - b_i & \leq 0 \quad \forall i = 1, \dots, m \end{aligned}$$

where $c \in \mathbb{R}^n, a_i \in \mathbb{R}^n, b_i \in \mathbb{R}$.

We can rephrase the problem as

$$\begin{aligned} - \max - c^T x & \\ \text{s.t.} \quad Ax & \leq b \end{aligned}$$

with $A = \begin{pmatrix} a_1^T \\ \vdots \\ a_m^T \end{pmatrix} \in \mathbb{R}^{m \times n}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$. A linear program is in *inequality standard form* if it is described as

$$\begin{aligned} \max c^T x & \\ \text{s.t.} \quad Ax & \leq b. \end{aligned}$$

By replacing $x \in \mathbb{R}^n$ with $x^+ - x^-$ for $x^+, x^- \geq \mathbf{0}$, this can be re-formulated as

$$\begin{aligned} \max c^T x^+ - c^T x^- & \\ \text{s.t.} \quad Ax^+ - Ax^- + s & = b \\ x^+, x^-, s \geq \mathbf{0}. & \end{aligned}$$

This shows that any linear program has a formulation in *equation standard form*

$$\begin{aligned} \min c^T x & \\ s.t. \, Ax & = b \\ x & \geq \mathbf{0} \end{aligned}$$

Dedication

We now come to our first example of linear programming being used in the world of finance. *Dedication* [1] is a technique to fund known liabilities in the future with a portfolio of assets whose yearly cash inflow matches the cash outflow of liabilities. Since this, liabilities will be paid off as they come due without the need to sell or buy assets in the future.
The portfolio is formed today and then held until all liabilities are paid off. Dedicated portfolios usually only consist of risk-free non-callable bonds since the portfolio future cash inflows need to be known when the portfolio is constructed.

According to [1], this technique is for example used by communes funds. To be able to eliminate the liabilities from the books, a commune can ask an investment bank for the minimum cost portfolio consisting of cash and bonds, such that all future liabilities can be paid off with the cash-flows from this portfolio.

The *maturity date* refers to the final payment date of a financial instrument (bond, option,...). The *face value* of bonds is the principal or redemption value. Interest payments are expressed as a percentage of the face value. Before reaching its maturity, the actual value of a bond may be greater or less than the face value, depending on the interest rate payable and the perceived risk of default. As bonds approach maturity, the actual value approaches the face value.
A *coupon* is the amount of interest paid per year expressed as a percentage of the face value of the bond or explicitely in liquidities. It is the interest rate that a bond issuer will pay to a bondholder.

Example

A bank recieves the following liability schedule:
We have 8 years

year	1	2	3	4	5	6	7	8
liability	12000	18000	20000	20000	16000	15000	12000	10000

We invest in bonds, each of face value of 100$. There are ten bonds:

bond	1	2	3	4	5	6	7	8	9	10
price	102	99	101	98	98	104	100	101	102	94
maturity year	1	2	2	3	4	5	5	6	7	8
coupon	5	3.5	5	3.5	4	9	6	8	9	7

All these bonds are widely available and can be purchased in any quantities at the stated price. The optimal bond portfolio consists of a certain amount of cash z_0 and x_i is the amount of bond i in the portfolio (we assume x_i does not need to be integral).

The parameters and the variables of the linear program :

- parameters
  - L_t=liability in year t.

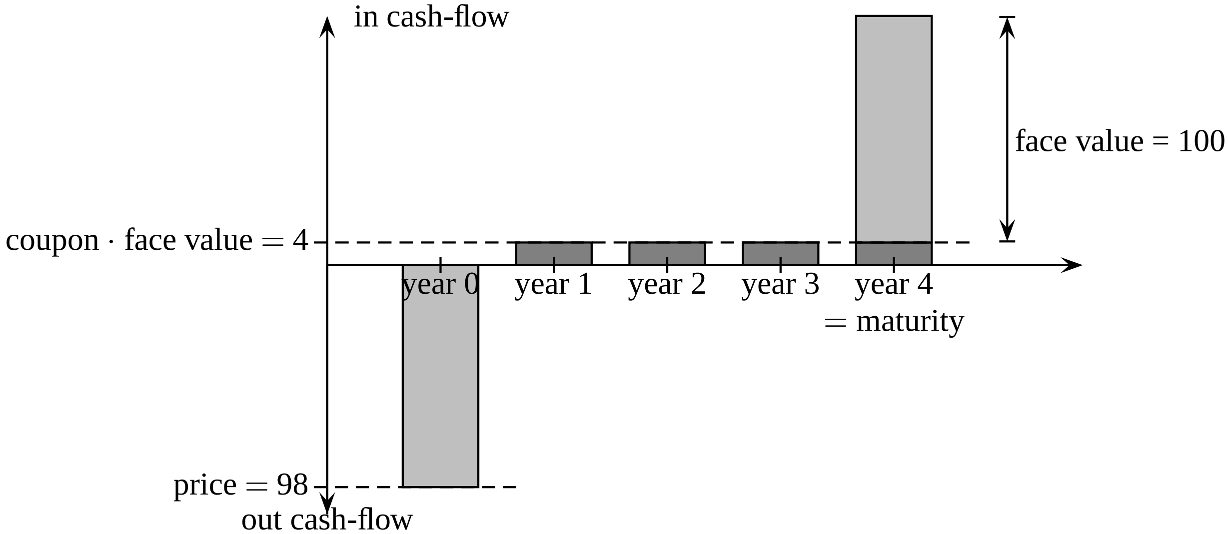

Figure 1: Cash-flow for Bond 5. The bond costs \$98 and pays back \$4 in Year 1, \$4 in Year 2, \$4 in Year 3 and \$100+\$4=\$104 in year 4.

  - p_i=price of bond i.
  - c_i is the annual coupon of bond i, and m_i is the maturity of bond i (in years).

- variables
  - x_i, the amount of bond i in the portfolio,
  - z_t=surplus at year t for $t = 0, \ldots, 8$.

We want to formulate and solve the linear program which finds the least cost portfolio of bonds to purchasetoday, to meet the liability schedule. The linear program is the following

$$\begin{array}{rcll} & \min \ z_0 + \sum_{i=1}^{10} x_i p_i & & \\ \text{s.t.} \quad z_{t-1} - z_t + \sum_{i:m_i=t} 100(x_i + c_i) + \sum_{i:m_i>t} x_i c_i & = & L_t & \forall 1 \leq t \leq 8 \\ x, z & \geq & \mathbf{0} & \end{array}$$

That is

$$\min \quad z_0 + 102x_1 + 99x_2 + 101x_3 + 98x_4 + 98x_5 + 104x_6 + 100x_7 + 101x_8 + 102x_9 + 94x_{10}$$

The first constraint is

$$12000 = z_0 - z_1 + 105x_1 + 3.5x_2 + 5x_3 + 3.5x_4 + 4x_5 + 9x_6 + 6x_7 + 8x_8 + 9x_9 + 7x_{10}$$

Does $z_8 = 0$ in the optimal solution ? (left as exercise).

Linear programming duality

We start with $\max\{c^T x \mid Ax \leq b\}$. For $\lambda \in \mathbb{R}^m_+$, the Lagrange dual is defined by

$$\begin{aligned} g(\lambda) &= \inf_{x \in \mathbb{R}^n} \{-c^T x + \lambda^T (Ax - b)\} \\ &= \inf_{x \in \mathbb{R}^n} \{(\lambda^T A - c^T) x\} - \lambda^T b \end{aligned}$$

If $\lambda^T A - c^T \neq 0$ then the infimum is $-\infty$, so the Lagrange dual problem is

$$\begin{aligned} \max \; -\lambda^T b & \\ \text{s.t.} \quad A^T \lambda &= c \\ \lambda &\geq \mathbf{0} \end{aligned}$$

By Lagrange duality,

$$\max\left\{-\lambda^T b \mid A^T \lambda = c, \lambda \in \mathbb{R}^m_+\right\} \leq \min\left\{-c^T x \mid Ax \leq b\right\}.$$

Equivalently,

$$\min\left\{\lambda^T b \mid A^T y = c, \lambda \in \mathbb{R}^m_+\right\} \geq \max\left\{c^T x \mid Ax \leq b\right\}$$

We refer to the left hand side as the dual linear program, and to the right hand side as the primal linear program. This inequality is called weak duality. In many cases, we can have equality.

Theorem 1 (**Strong Duality**). *If the primal problem is feasible and bounded, then the dual is feasible and bounded, and both have optimal solutions whose objective values coincide.*

Proof. If the primal is feasible and bounded, then an optimal solution exists (because we maximize a continuous function on a compact set [2]). Suppose that there exists an $x^* \in \mathbb{R}^n$ with $Ax^* < b$ (a Slater point), then by the duality theorem for convex optimization problems, we obtain

$$\max\{c^T x \mid Ax \leq b\} = \min\{b^T \lambda \mid A^T \lambda = c, \lambda \geq \mathbf{0}\}.$$

The case when a Slater point does not exist is left as an exercise. □

Now we get back to finance.

Pricing of call options

Arbitrage is a trading strategy which satisfies at least one of the following conditions

- A positive initial cash flow and no risk of loss later (type A).
  Ex: There is a security that pays off d at price p and another security that pays $2d$ at price $q < 2p$. Buy the $2d$ security and break it in equals pieces and sell them. Then you gain $2p - q > 0$.

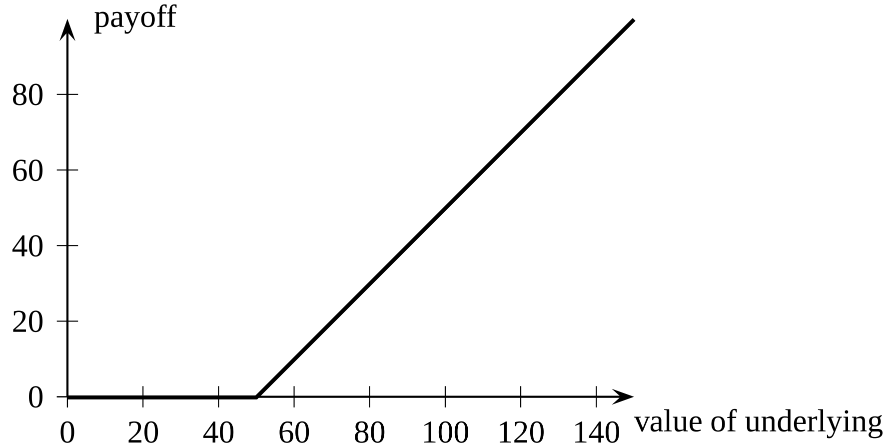

Figure 2: Payoff for a European call opion with strike price 50$.

- No initial cash input, no risk of loss, and a positive probability of making profits in the future (type B). Ex: Free lottery ticket.

European call option: At the expiration date the holder has the right to purchase a prescribed asset or underlying for prescribed amount (**strike price**). For example : Strike price 50$. The payoff depends on the cost of the asset. It is 0 if the cost of underlying is less than 50$, then increasing.

How should we price such a derivative ?
Simplifying assumption : s_0=current price of underlying. There are only two possible outcomes for the price in the future : $s_1^u = s_0 u$ and $s_1^d = s_0 d$ for $d < u$.
We denote the strike price by c_0.
Replication : Consider a portfolio of Δ shares of underlying and B cash.

- In the upstate, $\Delta s_0 u + BR$ where R is the risk-less interest rate.
- In the down state, $\Delta s_0 d + BR$.

So we define the payoff

- In the upstate : $c_1^u = \max\{s_0 u - c_0, 0\}$.
- In the down state : $c_1^d = \max\{s_0 d - c_0, 0\}$.

Solving

$$\Delta s_0 u + BR = c_1^u \text{ and } \Delta s_0 d + BR = c_1^d,$$

one obtains a solution

$$\Delta = \frac{c_1^u - c_1^d}{s_0(u-d)}$$

and

$$B = \frac{uc_1^d - dc_1^u}{R(u-d)}.$$

Since the portfolio is worth $s_0\Delta + B$ today and payoff of the derivate tomorrow.
Price for the derivate should be $s_0\Delta + B$.

References

[1] Gerard CORNUEJOLS & Reha TÜTÜNCÜ. Optimization Methods in Finance, *Cambridge University Press*, Cambridge (USA), 2007.

[2] Jacques DOUCHET & Bruno ZWAHLEN. Calcul Différentiel et Intégral, *PPUR*, 1990, pp 67.