Lecture 6: The Ellipsoid Method

27.10.2010

Lecturer: Prof. Friedrich Eisenbrand *Scribe: Nancy Moret*

Reduction of convex optimization problem

Recall that the Convex Optimization Problem (C.O.P) is the following:

$$\begin{array}{ll} \min & f_0(x) \\ s.th & f_i(x) \leq 0 \quad \forall i = 1, \ldots, m \\ & x \in \mathbb{R}^n \end{array}$$

where, $f_0, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ are convex functions.

This can be reduce to the decision problem, i.e, decide if the following set is feasible (not empty).

$$Q_\delta := \{x \in \mathbb{R}^n \,|\, f_i(x) \leq 0 \; \forall i = 1, \ldots, m \text{ and } f_0(x) \leq \delta\}$$

Binary search

Suppose $L_0 \leq p^* \leq U_0$ we can find p^* (approximately) by binary search, i.e,

1. Initialise $L = L_0, U = U_0$
2. Repeat
   $\delta := (U - L)/2$
   If $Q_\delta \neq \emptyset$ then $U := \delta$
   otherwise $L := \delta$

After k iterations, $L \leq p^* \leq U$ and $U - L = (U_0 - L_0)/2^k$.

Now we need to show that we can solve the decision problem. Assume, we are given the following:

- Bounded closed convex set $K \subseteq \mathbb{R}^n$
- $L > 0 : \text{vol}(K) \geq L$ (K is full-dimensional)
- $R > 0 : K \subseteq \{x \in \mathbb{R}^n \mid ||x|| \leq R\}$

- We must be able to solve separation problem. The separation problem is the following:

  For any $y \in \mathbb{R}^n$ we must be able to decide whether $y \in K$ or $y \notin K$
  and if $y \notin K$ return an hyperplane $c^T x = \beta$ such that $c^T \leq \beta, \forall x \in K$ and $c^T y > \beta$.

Definition 1. The *unit ball* in $\mathbb{R}^n$ is $B = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$. Let $f : \mathbb{R}^n \to \mathbb{R}^n : f(x) = Ax + b, A \in \mathbb{R}^{nn}$ regular, $b \in \mathbb{R}^n$. Then the set

$$\begin{aligned} E(A,b) &= f(B) \\ &= \{Ax+b | x \in B\} \\ &= \{y \in \mathbb{R}^n \mid ||A^{-1}(y-b)||^2 \leq 1\} \\ &= \{y \in \mathbb{R}^n \mid (A^{-1}(y-b)^T)A^{-1}(y-b) \leq 1\} \\ &= \{y \in \mathbb{R}^n \mid (y-b)^T(A^{-1})^T A^{-1}(y-b) \leq 1\} \end{aligned}$$

is termed an *ellipsoid*.

Note that $(A^{-1})^T A^{-1}$ is a positive definite, symmetric matrix.
The volume of an ellipsoid is:

$$\text{vol}(E(A,b)) = |\det(A)| \cdot \text{vol}(B) \cong |\det(A)| \cdot \frac{1}{\pi n}\left(\frac{2e\pi}{n}\right)^{\frac{n}{2}}$$

where $\frac{1}{\pi n}(\frac{2e\pi}{n})^{\frac{n}{2}}$ is an approximation of the volume of the unit ball.

The ellipsoid method

The goal for the *Ellipsoid method* is to find a point $b \in K$. This is done as follows: One iteratively computes ellipsoids that always contain the set K fully, such that the volume of the ellipsoids decreases from iteration to iteration.

The inital ellipsoid $E(A,b)$, is simply the ball of radius R, i.e, $A := RI, b = \mathbf{0}$.

(1) $E(A,b) = \{x \in \mathbb{R}^n \mid ||x|| \leq R\}$

(2) While $\text{vol}(E(A,b) \geq L$ Do

  (3) If $b \in K$ then RETURN b

  (4) Compute separating hyperplane $c^T x \leq \beta$
  (i.e. $c^T x \leq \beta$ $\forall x \in K$ and $c^T b > \beta$)

  (5) Compute $E(A',b') \supseteq E(A,b) \cap \{x | c^T x \leq \beta\}$

  (6) Update $E(A,b) := E(A',b')$

In order to show that the ellipsoid method works, we must assure that $\text{vol}(E(A',b'))$ is smaller than $\text{vol}(E(A,b))$.

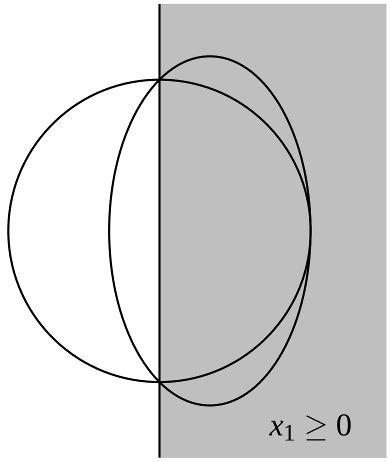

Figure 1: Unit ball and ellipsoid $E(A',b')$

Theorem 2. *For all $E(A,b) \subset \mathbb{R}^n$ and $c \in \mathbb{R}^n/\{\mathbf{0}\}$ there is an ellipsoid*

$$E(A',b') \supseteq E(A,b) \cap \{x | c^T x \leq c^T b\}$$

such that

$$\frac{vol(E(A',b'))}{vol(E(A,b))} \leq e^{-\frac{1}{2(n+1)}}$$

Without loss of generality we can suppose:

- $b = 0$ by applying a translation to the ellipsoid
- $E(A,b) = B$ by a linear transformation
- $c = (-1,0,\ldots,0)$ by rotation

Define

$$\begin{aligned} E := E(A',b') : &= E\left(\begin{pmatrix} \frac{n}{n+1} & 0 & \cdots & 0 \\ 0 & \sqrt{\frac{n^2}{n-1}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \sqrt{\frac{n^2}{n-1}} \end{pmatrix}, \begin{pmatrix} \frac{1}{n+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) \\ &= \left\{x \in \mathbb{R} \mid ||A'^{-1}(x-b)||^2 \leq 1\right\} \\ &= \left\{x \mid \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^n x_i^2 \leq 1\right\} \end{aligned}$$

Lemma 3. *One has $B \cap \{x | x_1 \geq 0\} \subseteq E$ and*

$$\frac{vol(E)}{vol(B)} \leq e^{-\frac{1}{2(n+1)}}$$

Proof. Let $x \in \mathbb{R}^n$ with $\|x\| \leq 1$ and $x_1 \geq 0$. Now we have:

$$\begin{aligned}
\|A'^{-1}(x-b)\|^2 &= \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \underbrace{\frac{n^2-1}{n}\sum_{i=2}^{n} x_i^2}_{\leq 1-x_1^2} \\
&\leq \left(\left(\frac{n+1}{n}\right)^2 - \frac{n^2-1}{n}\right)x_1^2 - \left(\frac{2}{n+1}\left(\frac{n+1}{n}\right)^2\right)x_1 + \left(\left(\frac{n+1}{n}\right)^2 \frac{1}{(n+1)^2} + \frac{n^2-1}{n^2}\right) \\
&= \frac{2n+2}{n^2}x_1^2 - \frac{2n+2}{n^2}x_1 + 1 =: f(x_1)
\end{aligned}$$

Since f is convex, the maximum of $f(x_1)$ for $0 \leq x_1 \leq 1$ must be attained for $x_1 \in \{0,1\}$. But $f(0) = 1 = f(1)$, hence indeed $B \cap \{x | x_1 \geq 0\} \subseteq E$.
The ratio of the volumes is:

$$\begin{aligned}
\frac{\mathrm{vol}(E)}{\mathrm{vol}(B)} = \|\det(A')\| &= \frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} && (1) \\
&= \left(1 - \frac{1}{n+1}\right)\left(1 + \frac{1}{n^2-1}\right)^{\frac{n-1}{2}} && (2) \\
&\leq e^{-\frac{1}{n+1}} e^{\frac{n-1}{2(n^2-1)}} && (3) \\
&= e^{-\frac{1}{n+1}} e^{\frac{n-1}{2(n-1)}} = e^{-\frac{1}{2(n+1)}} && (4)
\end{aligned}$$

For inequality (3) we use $1 + x \leq e^x \; \forall x \in \mathbb{R}$ and for inequality (4) we use $(n^2-1) = (n+1)(n-1)$. □

Lemma 4. *Let $0 \leq L \leq 1$, the ellipsoid method finds a feasible point after at most $k = 3n^2 \ln\left(\frac{2k}{L}\right)$ iterations.*

Proof. After $k := 2(n+1)n\ln\left(\frac{2R}{L}\right)$ iterations, we have:

$$\begin{aligned}
\mathrm{vol}(E(A,b)) &\leq R^n \underbrace{\mathrm{vol}(B)}_{\leq 2^n} \left(e^{-\frac{1}{2(n+1)}}\right)^k \\
&\leq (2R)^n \left(e^{-\frac{1}{2(n+1)}}\right)^k \\
&\leq (2R)^n e^{-n\ln\left(\frac{2R}{L}\right)} \\
&= (2R)^n \frac{L^n}{(2R)^n} \leq L
\end{aligned}$$

The last inequality comes from $L \leq 1$. □

Application to Mean Variance Optimization

Let us now outline, how the Ellipsoid method can be used to solve the Mean Variance Optimization problem. The goal is to find a point in

$$K := \left\{ x \in \mathbb{R}^n \,|\, \sum_{i=1}^{n} x_i = 1, x_i \geq 0, \forall i = 1,\ldots,n, \sum_{i=1}^{n} x_i \overline{r_i} \geq r, x^T Q x \leq \delta \right\}$$

- As starting bounds for the binary search, we can choose $L_0 := 0, U_0 := \max_{i,j} Q_{ij}$.
- Including ball: $R = 1$ (because $\sum_{i=1}^{n} x_i = 1$)
- The question is how find a hyperplane to separate $y \in \mathbb{R}^n$ from K, if $y \notin K$
  - If $y_i < 0$ return "$x_i \geq 0$",i.e, $c := -e_i, \beta := 0$
  - If $\sum_{i=1}^{n} y_i > 1$ return "$\sum_{i=1}^{n} x_i \leq 1$", i.e, $c := (1,\ldots,1), \beta := 1$
  - If $\sum_{i=1}^{n} y_i < 1$ return "$\sum_{i=1}^{n} x_i \geq 1$", i.e, $c := -(1,\ldots,1), \beta := -1$
  - If $y^T Q y > \delta$ return "$(Qy)^T x \leq \sqrt{\delta y^T Q y}$"
    It is a separating hyperplane because $y^T Q y > \delta \Rightarrow (Qy)^T y = y^T Q y > \sqrt{\delta y^T Q y}$.
    and $x^T Q x \leq \delta \Rightarrow (Qy)^T x = y^T Q x \leq \sqrt{y^T Q y \cdot x^T Q x} \leq \sqrt{y^T Q y \cdot \delta}$ using the Cauchy-Schwarz inequality.
- Unfortunately, K is not full-dimensional in this case, hence we can enlarge K to the following set with non-zero volume by relaxing the constraints:

$$K_\varepsilon := \left\{ x \in \mathbb{R}^n | 1 - \varepsilon \leq \sum_{i=1}^{n} x_i \leq 1 + \varepsilon_i,\ x_i \geq -\varepsilon,\ \sum_{i=1}^{n} \overline{r}_i x_i > r - \varepsilon,\ x^T Q x \leq \delta + \varepsilon \right\}.$$