Optimization Methods in Finance **(EPFL, Fall 2010)**

Lecture 2: Randomized weighted majority algorithm & zero sum games

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Lecturer: Prof. Friedrich Eisenbrand *Scribe: Miroslav Popovic*

Last time we examined few algorithms in order to help a forecaster to predict the future. For this he uses some experts and their visions of future. The goal is to be competitive with the best expert. To achieve this we presented a series of more and more powerful algorithms. For completeness we will repeat here the last one, the *Randomized Weighted Majority Algorithm*, and we will continue with its properties and possible applications.

Randomized Weighted Majority Algorithm

The setting is the same as before. There are N experts and time steps $t = 0, ..., T$. At time t, the forecaster randomly chooses an expert j. Nature reveals a loss vector $\ell^t \in [0,1]^N$ and the forecaster experiences the same loss ℓ_j^t as the chosen expert. Please note that in the last lecture we had an example where experts were predicting "YES" or "NO" (or 1 or 0). Here we have a generalization and consequently experts might experience any loss between 0 and 1.

The following question arises: *How to choose the experts in order to be competitive with those which are the best in terms of accumulated loss at the end?* The answer is: *At random*, where the probability distribution changes over time via multiplicative updates. If an expert experiences more loss, the probability to be chosen will be lower. The exact procedure is described in the following algorithm.

Algorithm 1 (Randomized Weighted Majority Algorithm)**.**

Initialize: Set $w_j := 1$ for $j = 1, ..., N$.
At time t: (Forecast)
Select expert with probability $p_j = \frac{w_j}{\sum_k w_k}$

(Observe and Re-Weight)
Observe loss vector ℓ^t
Forecaster experiences loss ℓ_j^t
For $j = 1, ..., N$
set $w_j := w_j (1-\varepsilon)^{\ell_j^t}$

Here $0 < \varepsilon < \frac{1}{2}$ is a fixed parameter.

Let $L^j = \sum_{t=0}^T \ell_j^t$ represent the accumulated loss of expert j over time. The forecaster's loss is a random variable with the following expected value:

$$E\left[\hat{L}^t\right] = \left(p^t\right)^T \ell^t, \tag{1}$$

where p^t denotes the probability distribution at time t (which expert is chosen) and ℓ^t is the loss vector at time t. The expected total loss of the forecaster during the whole process is $E\left[\hat{L}\right] = E\left[\hat{L}^0 + ... + \hat{L}^t\right]$.

Of course, after using the described algorithm for a while we would like to know our performance compared to the experts and the following theorem gives us the answer.

Theorem 1. *The expected total loss* $E\left[\hat{L}\right]$ *of a forecaster is bounded by*

$$E\left[\hat{L}\right] \leq \frac{\ln(N)}{\varepsilon} + (1+\varepsilon)L^j, (\forall j)\,. \tag{2}$$

Proof. One has

$$\begin{aligned}
W^{t+1} &= \sum_{j=1}^{N} w_j^{t+1} && (3)\\
&= \sum_{j=1}^{N} w_j^t (1-\varepsilon)^{\ell_j^t} && (4)\\
&\leq \sum_{j=1}^{N} w_j^t \left(1-\ell_j^t \varepsilon\right) && (5)\\
&= W^t - \sum_{j=1}^{N} w_j^t \ell_j^t \varepsilon && (6)\\
&= W^t - W^t \sum_{j=1}^{N} p_j^t \ell_j^t \varepsilon && (7)\\
&= W^t \left(1-\varepsilon E\left[\hat{L}_t\right]\right) && (8)\\
&\leq W^t e^{-\varepsilon E\left[\hat{L}_t\right]} && (9)
\end{aligned}$$

where W^{t+1} Equation (3)) denotes the total sum of experts' weights at time $t+1$, Equation (4) follows from the update rule defined in the Algorithm, Inequality (5) follows from $(1-\varepsilon)^x \leq (1-\varepsilon x)$ as seen in the exercise session, Equation (7) follows from the probability distribution of choosing expert j as defined in the Algorithm, Equation (8) follows from the Equation (1) and Inequality (9) follows from the fact that $1+x \leq e^x$.

If we continue by applying recursion derived in Equation (9) we have the following equations:

$$\begin{aligned}
W^{t+1} &\leq W^0 e^{-\varepsilon E\left[\hat{L}^0+\ldots+\hat{L}^t\right]} && (10)\\
&\leq N e^{-\varepsilon E\left[\hat{L}^0+\ldots+\hat{L}^t\right]} && (11)
\end{aligned}$$

On the other hand, from the way the update rule is defined in the algorithm, we have

$$W^{t+1} \geq (1-\varepsilon)^{L^j} \tag{12}$$

$$\Rightarrow (1-\varepsilon)^{L^j} \leq N e^{-\varepsilon E\left[\hat{L}\right]} \tag{13}$$

If we take ln of both sides:

$$L_j \ln(1-\varepsilon) \leq \ln(N) - \varepsilon E\left[\hat{L}\right] \tag{14}$$

$$\Leftrightarrow \varepsilon E\left[\hat{L}\right] \leq \ln(N) + \ln\left(\frac{1}{1-\varepsilon}\right) L^j \tag{15}$$

$$\leq \ln(N) + \left(\varepsilon + \varepsilon^2\right) L^j \tag{16}$$

$$\Leftrightarrow E\left[\hat{L}\right] = \frac{\ln(N)}{\varepsilon} + (1+\varepsilon)L^j, (\forall j) \tag{17}$$

Inequality (16) follows from the fact that $\ln\left(\frac{1}{1-\varepsilon}\right) \leq \varepsilon + \varepsilon^2$ as seen in the exercise session. □

Our next goal is to get as close as possible to the best expert by examining the average expert loss. For this we derive the following corollary.

Corollary 2. *Let $\delta > 0$ be an error parameter. With $\varepsilon = \min\left\{\frac{1}{2}, \frac{\delta}{2}\right\}$ and after $T = \frac{2\ln(N)}{\varepsilon\delta}$ time steps (rounds), the average expected loss satisfies*

$$\frac{E\left[\hat{L}\right]}{T} \leq \delta + \frac{L^j}{T}, (\forall j). \tag{18}$$

Proof. If we divide Equation (2) with T we have

$$\begin{aligned} \frac{E\left[\hat{L}\right]}{T} &\leq \frac{\ln(N)}{\varepsilon T} + (1+\varepsilon)\frac{L_j}{T} &(19)\\ &= \frac{\delta}{2} + \varepsilon\frac{L_j}{T} + \frac{L_j}{T} &(20)\\ &\leq \delta + \frac{L_j}{T} &(21) \end{aligned}$$

where Equation (20) follows from $T = \frac{2\ln(N)}{\varepsilon\delta}$ as defined in the Corollary and Equation (21) follows from $\varepsilon \leq \frac{\delta}{2}$ (Corollary - definition of ε) and the fact that $\frac{L_j}{T} \leq 1$ since $L^j = \sum_{t=0}^{T} \ell_j^t$ and $\ell_j^t \in [0,1]$. □

Zero-sum games

We continue with some notions of game theory which is an important field in economics and finance. At first, we define *return* and *rate of return*. If the price of an asset today is x_0 and tomorrow x_1, then the value $\frac{x_1}{x_0}$ is called the return and the value $\frac{x_1 - x_0}{x_0}$ is called the rate of return.

Example 3. *Suppose we can invest by following three different investment strategies (Stock A, Stock B and Money Market) and that three different scenarios are possible (Up, Stable and Down). The returns for each case are given in the following table.*

At this point several questions might arise. Is it possible that one loses money no matter which investment strategy is used? What is the optimal way to invest? Sometimes the answer might be quite obvious. For example, for the first question, one will not loose any money if he invests in Stock B since all returns in that column are greater than one.

Table 1: Returns for different scenarios.

	Stock A	Stock B	Money Market
Up	2	1.5	1
Stable	1.2	1.7	1.3
Down	0.8	1.2	1.4

The described problem is a zero-sum game. While one chooses columns, the nature chooses rows. More formal definition follows. In a *zero-sum game*, one is given a matrix $A \in \mathbb{R}^{m \times n}$ (payoff matrix) and we have two players: row player (Ralf) and column player (Celine). R chooses row i, C chooses column j and the payoff from Ralf to Celine is $A(i, j)$.

There are two kinds of possible strategies in zero-sum games.

- **Pure strategies** are those where Ralf picks a row i and Celine picks a column j. This yields payoff $A(i, j)$.
- **Mixed strategies** are those where we have a vector $Q \in \mathbb{R}^m_{\geq 0}$ such that $\sum_{i=1}^m Q_i = 1$ for Ralf and vector $P \in \mathbb{R}^n_{\geq 0}$ such that $\sum_{j=1}^n P_j = 1$ for Celine. The vectors Q and P model probability distributions. Q_i and P_j are probabilities of choosing row i and column j, respectively.

In the case of sequential play we have two possibilities depending on who plays first.

If Ralf plays first, he chooses mixed row strategy Q^*. After that, Celine chooses mixed column strategy P such that $Q^{*T} \cdot A \cdot P$ is maximal. Ralf wants to find Q^* which will minimize $\max_P Q^{*T} \cdot A \cdot P$. For this he needs to solve $\min_Q \max_P Q^{*T} \cdot A \cdot P$.

Likewise, if Celine plays first, she wants to solve $\max_P \min_Q Q^T \cdot A \cdot P$.

There is a question how we can compare last two values. Intuitively, the amount that Ralf has to pay to Celine should be not be larger, if Ralf is allowed play second, i.e.

$$\min_Q \max_{P:=P(Q)} Q^T \cdot A \cdot P \geq \max_P \min_{Q:=Q(P)} Q^T \cdot A \cdot P \tag{22}$$

Here $P(Q)$ denotes that the strategy P is allowed to depend on Q (i.e. in the left hand side expression, Celine chooses her strategy *after* Ralf).

Inequality (22) can be formally shown as follows: Let Q_1 be the mixed strategy that Ralf would choose if he would have to play first and P_1 be the strategy with which Celine answers. Similarly, let P_2 be the mixed strategy that Celine would play, if she had to play first and Q_2 Ralfs best response to that. Then

$$Q_1AP_1 \geq Q_1AP_2 \geq Q_2AP_2$$

Note that Inequality (22) holds for P, Q being mixed strategies and for the case, when both are restricted to be pure strategies. However, an even stronger claim than what is expressed in the Inequality (22) can be made and it is expressed in the following Minimax Theorem (which only holds for mixed strategies).

Theorem 4. *(Minimax Theorem) For a two-person, zero-sum game there exists a mixed strategy for each player, such that no matter who plays first the best possible payoff for both players is the same value, i.e.*

$$\min_Q \max_P Q^T \cdot A \cdot P = \max_P \min_Q Q^T \cdot A \cdot P, \tag{23}$$

where Q and P range over mixed row and column strategies, respectively.