DISCRETE OPTIMIZATION
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Spring 2010
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Sample exam

Last name: First name:

Exercise:	1	2	3	4	5	6	7	8	Σ
max points:	10	10	10	10	10	10	10	10	50
achieved points:									
chosen exercises:									

Check whether the exam is complete: It should have 9 pages (Exercises 1–8). Write your name on the title page. Solutions have to be written below the exercises. Solutions must be comprehensible. In case of lack of space, you can ask for additional paper from the exam supervision. Please put your name on each additional sheet and indicate which exercise it belongs to.

Use neither pencil nor red colored pen!

Duration: 120 min

Grading:
Every exercise gives 10 points, and you are supposed to solve 5 of them. There are 6 exercises marked with [$*$] and two exercises marked with [Δ]. Math students can choose among the [$*$]-exercises. Non-math students can choose among all exercises. **Please mark the 5 exercises you have chosen in the tabular above!**

You are allowed to bring a pocket calculator and an A4-"cheat-sheet".

Exercise 1 [$*$]: (Multiple Choice, points $\{-1,0,1\}$ each)
No justifications needed. Mark 'yes' or 'no'. **Wrong answers cause negative points!**

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a) A set $C \subseteq \mathbb{R}^n$ is convex if and only if $\lambda x + (1-\lambda)y \in C$ for any $x, y \in C$ and $\lambda \in \mathbb{R}$.

○ yes ○ **<u>no</u>**

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b) One has

$$\min\{c^T x \mid Ax = b,\ x \geq \mathbf{0}\} = \max\{b^T y \mid A^T y \leq c\}$$

given that both linear programs are feasible ($A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$).

○ **<u>yes</u>** ○ no

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c) Given a linear program

$$\max\{c^T x \mid Ax \leq b\},$$

with $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. If the LP is feasible and bounded, then there is a *roof* B such that its vertex is an optimal solution to the LP.

○ yes ○ **<u>no</u>**

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d) Given a matrix $A \in \mathbb{Z}^{m\times n}$ with $m \geq 2$. Let $A' \in \mathbb{Z}^{m\times n}$ be a matrix obtained from A by the *elementary* row operation of adding an integer multiple of row 1 to row 2. A is totally unimodular if and only if A' is totally unimodular.

○ yes ○ **<u>no</u>**

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e) Given a linear program

$$\max\{c^T x \mid Ax \leq b\},$$

with $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. If the LP is feasible and bounded, then there is a an optimal solution x to the LP such that at most m entries of x are nonzero.

○ **<u>yes</u>** ○ no

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f) Given a linear program

$$\max\{c^T x \mid Ax \leq b\},$$

with $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. If $x^{(1)}$ and $x^{(2)}$ are optimal solutions for the LP, then every vector $x \in \text{conv}(x^{(1)}, x^{(2)})$ is optimal.

○ **<u>yes</u>** ○ no

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g) For any graph $G = (V, E)$, its node-edge incidence matrix is totally unimodular.

○ yes ○ **<u>no</u>**

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h) Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$. If there is a $\lambda \in \mathbb{R}^m$ such that $A^T\lambda \geq 0$ and $b^T\lambda < 0$, then the system $Ax = b$, $x \geq 0$ is infeasible.

○ **<u>yes</u>** ○ no

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i) Given a directed graph $G = (V, A)$ and a node $v \in V$, a shortest path tree rooted in v can be computed in time $O(|V| + |A|)$.

○ **<u>yes</u>** ○ no

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j) There is a linear program

$$\max\{c^T x \ :\ Ax \leq b\},$$

with $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ such that both the LP and its dual are infeasible.

○ **<u>yes</u>** ○ no

Exercise 2 [$*$] (LP duality):

Consider the following linear program:

$$
\begin{array}{lrcrcrcr}
\min & 2x_1 & + & 2x_2 & + & 4x_3 & & \\
 & x_1 & + & 2x_2 & + & 4x_3 & = & 20 \\
 & -x_1 & & & + & 3x_3 & \leq & 10 \\
 & & - & 2x_2 & + & x_3 & \geq & 3 \\
 & 4x_1 & + & x_2 & & & \leq & 40 \\
 & x_1 & - & 10x_2 & + & x_3 & \geq & -3
\end{array} \tag{1}
$$

(a) Transform LP (1) to (inequality) standard form.

(b) Write down a dual of the LP in standard form.

(c) Show that $x^* := (\frac{19}{8}, \frac{9}{16}, \frac{33}{8})^T$ is an optimal solution for the LP (1) by giving a suitable solution for the dual LP (Hint: Use the complementary slackness theorem you have seen in the exercises: Given an optimal solution x^* for the primal, there is an optimal solution y^* for the dual such that $y_i^* = 0$ for all rows i of the primal that are not satisfied with equality by x^*).

Solution:

(a) The LP in inequality standard form looks as follows:

$$\begin{array}{lrcrcrcr} \max & -2x_1 & - & 2x_2 & - & 4x_3 & & \\ & x_1 & + & 2x_2 & + & 4x_3 & \leq & 20 \\ & -x_1 & - & 2x_2 & - & 4x_3 & \leq & -20 \\ & -x_1 & & & + & 3x_3 & \leq & 10 \\ & & & 2x_2 & - & x_3 & \leq & -3 \\ & 4x_1 & + & x_2 & & & \leq & 40 \\ & -x_1 & + & 10x_2 & - & x_3 & \leq & 3 \end{array}$$

(b) A dual is:

$$\begin{array}{lrcrcrcrcrcrcr} \min & 20y_1 & - & 20y_2 & + & 10y_3 & - & 3y_4 & + & 40y_5 & + & 3y_6 & & \\ & y_1 & - & y_2 & - & y_3 & & & + & 4y_5 & - & y_6 & = & -2 \\ & 2y_1 & - & 2y_2 & & & + & 2y_4 & + & y_5 & + & 10y_6 & = & -2 \\ & 4y_1 & - & 4y_2 & + & 3y_3 & - & y_4 & & & - & y_6 & = & -4 \\ & y_1 & , & y_2 & , & y_3 & , & y_4 & , & y_5 & , & y_6 & \geq & 0 \end{array}$$

(c) We see that x^* is a feasible solution as all constraints are satisfied. The first four constraints are satisfied with equality, whereas the last 2 constraints are satisfied with strict inequality. From the complementary slackness theorem we know that if x^* is optimal, then the dual has an optimal solution where $y_5 = y_6 = 0$.

Thus consider the system

$$\begin{array}{rcrcrcrcr} y_1 & - & y_2 & - & y_3 & & & = & -2 \\ 2y_1 & - & 2y_2 & & & + & 2y_4 & = & -2 \\ 4y_1 & - & 4y_2 & + & 3y_3 & - & y_4 & = & -4 \end{array}$$

The first two columns are linearly dependent, thus we omit column 2 and consider the system:

$$\begin{array}{rcrcrcr} y_1 & - & y_3 & & & = & -2 \\ 2y_1 & & & + & 2y_4 & = & -2 \\ 4y_1 & + & 3y_3 & - & y_4 & = & -4 \end{array}$$

We see that $y_1 = -\frac{11}{8}$, $y_3 = \frac{5}{8}$ and $y_4 = \frac{3}{8}$ is the solution to the system. This implies that $y = (0, \frac{11}{8}, \frac{5}{8}, \frac{3}{8}, 0, 0)$ is a feasible solution of the dual. Its objective value is $-\frac{179}{8}$ as the objective value of x^* for the primal. This asserts that x^* is optimal.

Use reverse side if you need more space

Exercise 3 [Δ] (IP modeling):

Consider the following transportation problem: F is a set of warehouses that are owned by our company, G is a set of different goods and C is a set of clients (all sets are finite). Let $s_{ij} \geq 0$ be the amount of good $i \in G$, that is available in warehouse $j \in F$. Furthermore $d_{ik} \geq 0$ denotes the amount of good $i \in G$, that client $k \in C$ requests. It costs $c_{ijk} \geq 0$ to transport one unit of good $i \in G$ from warehouse $j \in F$ to customer $k \in C$. All quantities are integer. (We assume that the costs grow linear with the amount and goods are splittable in integer quantities). Formulate an integer program that determines the cheapest way to transport the goods to the clients such that: The demand of each client is satisfied and the supplies of the warehouses are not exceeded. Explain the meaning of the variables you used.

Is the polyhedron of the linear programming relaxation integral? Justify your answer by giving an argument why it is integer, or give a counterexample if it is not integer.

Solution:
We use decision variables

$$x_{ijk} = \text{amount of good } i \in G \text{ which is transported from } j \in F \text{ to } k \in C$$

Then the desired IP is

$$\begin{aligned}
\min \sum_{i\in G}\sum_{j\in F}\sum_{k\in C} c_{ijk}x_{ijk} & & & \\
\sum_{k\in C} x_{ijk} &\leq s_{ij} & \forall i \in G\, \forall j \in F & \quad \text{(supply not exceeded)}\\
\sum_{j\in F} x_{ijk} &\geq d_{ik} & \forall i \in G\, \forall k \in C & \quad \text{(demand fulfilled)}\\
x_{ijk} &\geq 0 & \forall i \in G\, \forall j \in F\, \forall k \in C & \quad \text{(non-negativity)}\\
x &\in \mathbb{Z} & &
\end{aligned}$$

Note that every variable x_{ijk} appear in exactly one of the supply constraints, and in exactly one of the demand constraints. Now consider the LP relaxation, and consider the matrix A corresponding to the supply and demand constraints. Since each x_{ijk} appears exactly in one supply and one demand constraint, we have that each column of the matrix has exactly two 1-entries. Moreover, one of the entries belongs to a "supply row" and the other one belongs to the "demand row". Hence A has the structure of a node-edge incidence matrix of a bipartite graph. Thus A is totally unimodular. The full matrix of the LP relaxation is of the form

$$B := \begin{pmatrix} A \\ I_n \end{pmatrix}$$

where $n = |F| \cdot |G| \cdot |C|$. Hence B is totally unimodular as well, and we conclude that the linear programming relaxation is integral.

Use reverse side if you need more space

Exercise 4 [$*$] (Roofs):

Consider a linear program

$$\max\{c^T x \;:\; Ax \leq b\}$$

with $A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ c \in \mathbb{R}^n$.
Let $B \subseteq \{1,\ldots,m\}$ be a subset of row indexes of A such that $|B| = n$ and A_B has full rank. Show that if $c \in \text{cone}(a_i \;:\; i \in B)$, then B is a *roof*

Solution:
A solution is given in the lecture notes in the proof of Lemma 3.2.

Use reverse side if you need more space

Exercise 5 [$*$] (Simplex algorithm):

Consider the following LP:

$$\begin{array}{lccccccc} \max & 2y_1 & + & 2y_2 & + & 4y_3 & & \\ & y_1 & - & 2y_2 & + & 2y_3 & \leq & -1 \\ & 3y_1 & - & 2y_2 & + & 4y_3 & \leq & -3 \\ & y_1 & & & & & \leq & 0 \\ & y_2 & & & & & \leq & 0 \\ & y_3 & & & & & \leq & 0 \end{array}$$

Solve the LP using the simplex method.
Start with the roof $B = \{3,4,5\}$.
For each iteration of the simplex method, the violated constraint that should enter the roof, the constraint that has to leave the roof, the new roof and its vertex.
Also write down an optimal solution and its value. On the next page, you find the inverse matrices for all possible roofs.

Solution:
Let A be the matrix of the LP. We start with the roof $B = \{3,4,5\}$. The corresponding vertex solution is $y = (0,0,0)^T$. The conic combination of c is $c = \mu_1 \cdot a_3 + \mu_2 \cdot a_4 + \mu_3 \cdot a_5$ with $\mu = (2,2,4)$.
Both constraints 1 and 2 are violated. We choose 1 to enter the roof. Hence we compute a solution to the system

$$\lambda_1 a_3 + \lambda_2 a_4 + \lambda_3 a_5 + a_1 = 0,$$

which is $\lambda = (-1,2,-2)$. We have $-\frac{\mu_1}{\lambda_1} = 2 = -\frac{\mu_3}{\lambda_3}$, thus we can choose whether 3 or 5 leaves the basis. We choose 3. Hence we get the new conic combination $c = \mu_1 \cdot a_1 + \mu_2 \cdot a_4 + \mu_3 \cdot a_5$ with $\mu = (2,6,8)$ and the new roof is $B = \{1,4,5\}$ with vertex solution $y = (-1,0,0)^T$.
This solution satisfies all constraints, thus it is optimal. The objective value is -2.

Use reverse side if you need more space

- $B := \{1,2,4\}, A_B^{-1} = \begin{pmatrix} -2 & 1 & 2 \\ 0 & 0 & 1 \\ \frac{3}{2} & -\frac{1}{2} & 2 \end{pmatrix}$.
- $B := \{1,3,4\}, A_B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$.
- $B := \{1,4,5\}, A_B^{-1} = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
- $B := \{2,4,5\}, A_B^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
- $B := \{3,4,5\}, A_B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Exercise 6 [$*$] (Total unimodularity):

An *interval matrix* is a matrix $M \in \{0,1\}^{m\times n}$ where in each row, the 1-entries appear as a consecutive block. I.e. for each row i we have

$$\forall j,k,\ell \quad \text{with } j \le k \le \ell: \quad \text{If } M(i,j)=1 \text{ and } M(i,\ell)=1, \text{ then } M(i,k)=1.$$

Prove that M is totally unimodular. (Hint: Elementary column operations might help)

Solution:

Note that interval matrices are closed under taking submatrices. Thus it is sufficient to show that if M is a square matrix (i.e. $n=m$), then: $\det(M) \in \{-1,0,1\}$.

We perform the following elementary column operations: For each $i = n-1, n-2, \dots, 1$, we subtract column i from column $i+1$. Let M' be the resulting matrix. Clearyly $\det(M) = \det(M')$. Note that M' has at most one 1 entry and at most one -1 entry per row, all other entries are 0.

Otherwise, while there is a row with only one nonzero entry, we perform laplace expansion along that row, until a submatrix M'' remains with $|\det(M'')| = |\det(M')|$. If M'' contains a zero-row, then $\det(M') = 0$ and we are done. Otherwise, every row has exactly one 1 and one -1 entry. Thus the sum of the columns is the zero vector, hence $\det(M'') = 0$.

Use reverse side if you need more space

Exercise 7 [$*$] (Vertices):

Let $P = \{x \in \mathbb{R}^n \colon Ax \le b\}$ be a polyhedron and let $x^* \in P$. You can assume that A is of full column rank. Show that x^* is a vertex of P if and only if there exists a set $B \subseteq \{1,\dots,m\}$ such that $|B| = n$, A_B is invertible and $A_B x^* = b_B$. Here the matrix A_B and the vector b_B consists of the rows of A indexed by B and the components of b indexed by B respectively.

Solution:

"$\Rightarrow$":

Let $H := \{x \in \mathbb{R}^n : c^T x = \beta\}$ be the hyperplane that defines the vertex x^*, i.e. $H \cap P = \{x^*\}$, and $c^T x \le \beta$ for all $x \in P$. Now consider the linear program $\max\{c^T x : Ax \le b\}$. By construction, x^* is the unique optimal solution for this LP. On the other hand, since the LP is bounded and of full column rank, the simplex algorithm will compute an optimal solution that is a vertex of a roof. Thus x^* *is* the vertex of a roof, and (ii) follows.

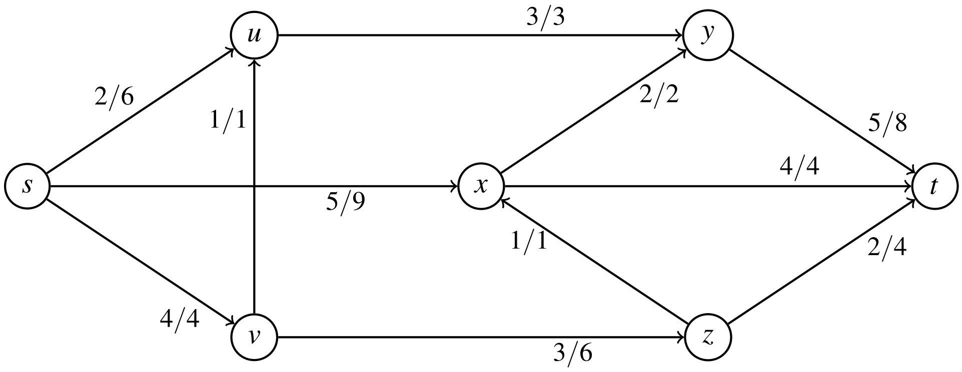
"$\Leftarrow$":

Note that x^* is the vertex of the roof A_B. Let $c \in \operatorname{cone}(A_B)$ be a *strictly positive* conic combination of the rows of A_B. As seen in Exercise 5 on Sheet 2, the vertex x^* of the roof A_B is then a unique optimal solution to the linear program $\max\{c^T x : A_B x \le b_B\}$. Since $x^* \in P$, this implies that x^* is the unique optimal solution to the linear program $\beta := \max\{c^T x : Ax \le b\}$. Hence, $c^T x \le \beta$ for all $x \in P$ and $\{x \in \mathbb{R}^n : c^T x = \beta\} \cap P = x^*$. Thus x^* is a vertex.

Use reverse side if you need more space

Exercise 8 [Δ] (Max $s-t$-flows):

Consider the following graph $G = (V,A)$. The labels on the arcs $a \in A$ are of the form $f(a)/u(a)$, i.e. they define functions $f: A \to \mathbb{Q}_{\geq 0}$ and $u: A \to \mathbb{Q}_{\geq 0}$.



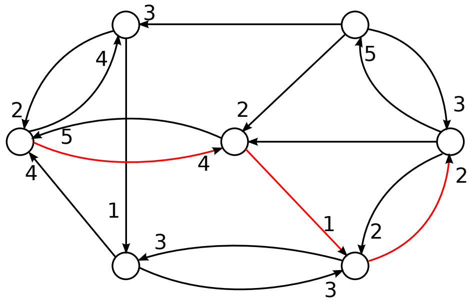
(a) Argue why f is a *feasible $s-t$-flow* in G subject to the capacities u. What is the value of the flow?

(b) Perform the Ford-Fulkerson algorithm to compute a maximum $s-t$-flow in G. For each iteration give the residual network. You can start with the flow f. Give the flow, its value and a minimum $s-t$ cut.

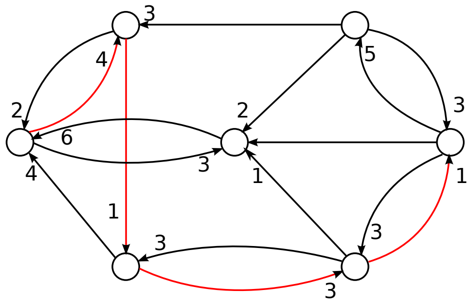
Solution:

(a) f is a *flow* since for all $v \in V - s,t$ we have $f(\delta^{in}(v)) = f(\delta^{in}(v))$. f is feasible since $0 \leq f(a) \leq u(a)$ for each $a \in A$.

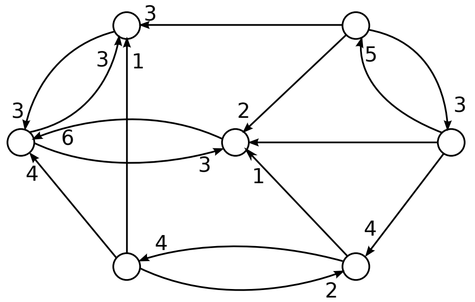
(b) We start with the residual network corresponding to f.



An $s-t$-path is marked red. After augmenting we obtain the following residual network:



An $s-t$-path is marked red. After augmenting we obtain the following residual network:



s and t are not connected anymore. Thus we found the optimal $s-t$-flow f given as

$$f(s,u) = 3,\ f(s,v) = 4,\ f(s,x) = 6,\ f(u,y) = 3,\ f(v,u) = 0,\ f(v,z) = 4,$$

$$f(x,y) = 2,\ f(x,z) = 0,\ f(x,t) = 4,\ f(y,t) = 5,\ f(z,t) = 4.$$

of value 13. A min cut is given by the nodes $\{s,u,x\}$.

Use reverse side if you need more space