DISCRETE OPTIMIZATION
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Spring 2010
06.05.2010

Sample exam

Last name: First name:

Exercise:	1	2	3	4	5	6	7	8	Σ
max points:	10	10	10	10	10	10	10	10	50
achieved points:									
chosen exercises:									

Check whether the exam is complete: It should have 9 pages (Exercises 1–8). Write your name on the title page. Solutions have to be written below the exercises. Solutions must be comprehensible. In case of lack of space, you can ask for additional paper from the exam supervision. Please put your name on each additional sheet and indicate which exercise it belongs to.

Use neither pencil nor red colored pen!

Duration: 120 min

Grading:
Every exercise gives 10 points, and you are supposed to solve 5 of them. There are 6 exercises marked with [$*$] and two exercises marked with [Δ]. Math students can choose among the [$*$]-exercises. Non-math students can choose among all exercises. **Please mark the 5 exercises you have chosen in the tabular above!**

You are allowed to bring a pocket calculator and an A4-"cheat-sheet".

Exercise 1 [$*$]: (Multiple Choice, points $\{-1, 0, 1\}$ each)
No justifications needed. Mark 'yes' or 'no'. **Wrong answers cause negative points!**

a)	A set $C \subseteq \mathbb{R}^n$ is convex if and only if $\lambda x + (1-\lambda)y \in C$ for any $x, y \in C$ and $\lambda \in \mathbb{R}$.	○ yes ○ no
b)	One has $$\min\{c^T x \mid Ax = b,\ x \geq \mathbf{0}\} = \max\{b^T y \mid A^T y \leq c\}$$ given that both linear programs are feasible ($A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$).	○ yes ○ no
c)	Given a linear program $$\max\{c^T x \mid Ax \leq b\},$$ with $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. If the LP is feasible and bounded, then there is a *roof* B such that its vertex is an optimal solution to the LP.	○ yes ○ no
d)	Given a matrix $A \in \mathbb{Z}^{m\times n}$ with $m \geq 2$. Let $A' \in \mathbb{Z}^{m\times n}$ be a matrix obtained from A by the *elementary* row operation of adding an integer multiple of row 1 to row 2. A is totally unimodular if and only if A' is totally unimodular.	○ yes ○ no
e)	Given a linear program $$\max\{c^T x \mid Ax \leq b\},$$ with $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. If the LP is feasible and bounded, then there is a an optimal solution x to the LP such that at most m entries of x are nonzero.	○ yes ○ no
f)	Given a linear program $$\max\{c^T x \mid Ax \leq b\},$$ with $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. If $x^{(1)}$ and $x^{(2)}$ are optimal solutions for the LP, then every vector $x \in \mathrm{conv}(x^{(1)}, x^{(2)})$ is optimal.	○ yes ○ no
g)	For any graph $G = (V, E)$, its node-edge incidence matrix is totally unimodular.	○ yes ○ no
h)	Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$. If there is a $\lambda \in \mathbb{R}^m$ such that $A^T\lambda \geq 0$ and $b^T\lambda < 0$, then the system $Ax = b$, $x \geq 0$ is infeasible.	○ yes ○ no
i)	Given a directed graph $G = (V, A)$ and a node $v \in V$, a shortest path tree rooted in v can be computed in time $O(\|V\| + \|A\|)$.	○ yes ○ no
j)	There is a linear program $$\max\{c^T x\ :\ Ax \leq b\},$$ with $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ such that both the LP and its dual are infeasible.	○ yes ○ no

Exercise 2 [$*$] (LP duality):

Consider the following linear program:

$$\begin{array}{lrcrcrcr} \min & 2x_1 & + & 2x_2 & + & 4x_3 & & \\ & x_1 & + & 2x_2 & + & 4x_3 & = & 20 \\ & -x_1 & & & + & 3x_3 & \leq & 10 \\ & & - & 2x_2 & + & x_3 & \geq & 3 \\ & 4x_1 & + & x_2 & & & \leq & 40 \\ & x_1 & - & 10x_2 & + & x_3 & \geq & -3 \end{array} \tag{1}$$

(a) Transform LP (1) to (inequality) standard form.

(b) Write down a dual of the LP in standard form.

(c) Show that $x^* := (\frac{19}{8}, \frac{9}{16}, \frac{33}{8})^T$ is an optimal solution for the LP (1) by giving a suitable solution for the dual LP (Hint: Use the complementary slackness theorem you have seen in the exercises: Given an optimal solution x^* for the primal, there is an optimal solution y^* for the dual such that $y_i^* = 0$ for all rows i of the primal that are not satisfied with equality by x^*).

Solution:

Use reverse side if you need more space

Exercise 3 [Δ] (IP modeling):

Consider the following transportation problem: F is a set of warehouses that are owned by our company, G is a set of different goods and C is a set of clients (all sets are finite). Let $s_{ij} \geq 0$ be the amount of good $i \in G$, that is available in warehouse $j \in F$. Furthermore $d_{ik} \geq 0$ denotes the amount of good $i \in G$, that client $k \in C$ requests. It costs $c_{ijk} \geq 0$ to transport one unit of good $i \in G$ from warehouse $j \in F$ to customer $k \in C$. All quantities are integer. (We assume that the costs grow linear with the amount and goods are splittable in integer quantities). Formulate an integer program that determines the cheapest way to transport the goods to the clients such that: The demand of each client is satisfied and the supplies of the warehouses are not exceeded. Explain the meaning of the variables you used.

Is the polyhedron of the linear programming relaxation integral? Justify your answer by giving an argument why it is integer, or give a counterexample if it is not integer.

Solution:

Use reverse side if you need more space

Exercise 4 [$*$] (Roofs):

Consider a linear program

$$\max\{c^T x \,:\, Ax \leq b\}$$

with $A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ c \in \mathbb{R}^n$.
Let $B \subseteq \{1,\ldots,m\}$ be a subset of row indexes of A such that $|B| = n$ and A_B has full rank. Show that if $c \in \text{cone}(a_i \,:\, i \in B)$, then B is a *roof*

Solution:

Use reverse side if you need more space

Exercise 5 [$*$] (Simplex algorithm):

Consider the following LP:

$$
\begin{array}{lrcrcrcr}
\max & 2y_1 & + & 2y_2 & + & 4y_3 & & \\
 & y_1 & - & 2y_2 & + & 2y_3 & \leq & -1 \\
 & 3y_1 & - & 2y_2 & + & 4y_3 & \leq & -3 \\
 & y_1 & & & & & \leq & 0 \\
 & y_2 & & & & & \leq & 0 \\
 & y_3 & & & & & \leq & 0
\end{array}
$$

Solve the LP using the simplex method.
Start with the roof $B = \{3, 4, 5\}$.
For each iteration of the simplex method, the violated constraint that should enter the roof, the constraint that has to leave the roof, the new roof and its vertex.
Also write down an optimal solution and its value. On the next page, you find the inverse matrices for all possible roofs.

Solution:

Use reverse side if you need more space

- $B := \{1,2,4\}, A_B^{-1} = \begin{pmatrix} -2 & 1 & 2 \\ 0 & 0 & 1 \\ \frac{3}{2} & -\frac{1}{2} & 2 \end{pmatrix}.$
- $B := \{1,3,4\}, A_B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.$
- $B := \{1,4,5\}, A_B^{-1} = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$
- $B := \{2,4,5\}, A_B^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$
- $B := \{3,4,5\}, A_B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

Exercise 6 [$*$] (Total unimodularity):

An *interval matrix* is a matrix $M \in \{0,1\}^{m\times n}$ where in each row, the 1-entries appear as a consecutive block. I.e. for each row i we have

$$\forall j,k,\ell \quad \text{with } j \le k \le \ell: \quad \text{If } M(i,j) = 1 \text{ and } M(i,\ell) = 1, \text{ then } M(i,k) = 1.$$

Prove that M is totally unimodular. (Hint: Elementary column operations might help)

Solution:

Use reverse side if you need more space

Exercise 7 [$*$] (Vertices):

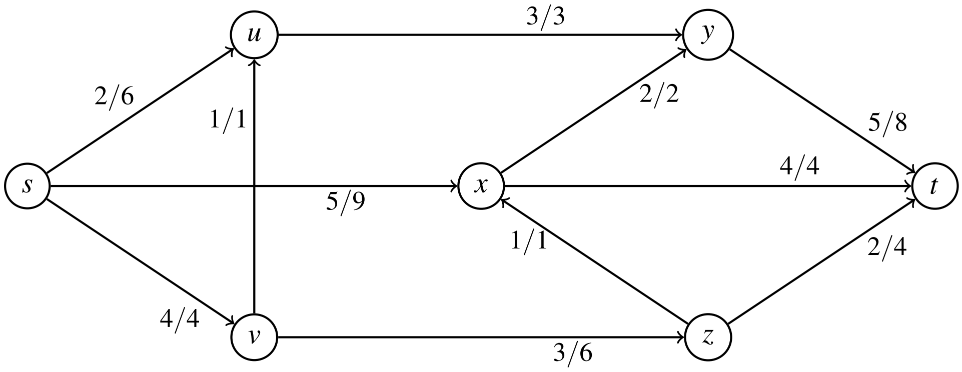
Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron and let $x^* \in P$. You can assume that A is of full column rank. Show that x^* is a vertex of P if and only if there exists a set $B \subseteq \{1, \ldots, m\}$ such that $|B| = n$, A_B is invertible and $A_B x^* = b_B$. Here the matrix A_B and the vector b_B consists of the rows of A indexed by B and the components of b indexed by B respectively.

Solution:

Use reverse side if you need more space

Exercise 8 [Δ] (Max $s-t$-flows):

Consider the following graph $G=(V,A)$. The labels on the arcs $a \in A$ are of the form $f(a)/u(a)$, i.e. they define functions $f: A \to \mathbb{Q}_{\geq 0}$ and $u: A \to \mathbb{Q}_{\geq 0}$.



(a) Argue why f is a *feasible $s-t$-flow* in G subject to the capacities u. What is the value of the flow?

(b) Perform the Ford-Fulkerson algorithm to compute a maximum $s-t$-flow in G. For each iteration give the residual network. You can start with the flow f. Give the flow, its value and a minimum $s-t$ cut.

Solution:

Use reverse side if you need more space