Fall 2014 Randomized Algorithms	Deadline: Dec 17, 2014

Problem Set 4

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Note: Problems 6 and 7 in this set are bonus problems. The points for these problems, if you attempt it, will be added to your final score over all problem sets. You may also skip it as the maximum points for this problem set is only 24 points.

1. Suppose $11n$ points are placed around a circle and colored with n different colors in such a way that each color is applied to exactly 11 points. Show that in any such coloring, there must be a set of n points containing one point of each color but not containing any pair of adjacent points. **[4 pts]**

2. Use the probabilistic method to show that an expanding bipartite graph (with bipartitions L and R) with the following properties exists for n sufficiently large:

   - $|L| = |R| = n$.
   - Every vertex in L has degree $n^{3/4}$ and every vertex in R has degree at most $3n^{3/4}$.
   - Every subset of $n^{3/4}$ vertices in L has at least $n - n^{3/4}$ neighbours in R.

   [4 pts]

3. For a suitably small function $f(n)$ and large enough even integer n, show that there exists a graph $G(V, E)$ with $|V| = n$ such that for every subset $T \subseteq V$ of size $n/2$,
$$\left|\delta(T) - \frac{n^2}{8}\right| \leq f(n).$$
   How small can you make the function $f(n)$? **[4 pts]**

4. 2SAT the problem of determining whether a given boolean formula (in conjunctive normal form) where each clause comprises of exactly two literals is satisfiable or not. Consider the following randomized algorithm for deciding a 2SAT formula:

   - Start with an arbitrary assignment to the variables.
   - While there is a clause that is unsatisfied flip the value of one of the variables in the unsatisfied clause.

Assume that each clause contains literals from distinct variables. What is the expected number of steps for the above algorithm to find a satisfying assignment assuming that the input is a satisfiable 2SAT formula? Transform this into a one-sided error Monte Carlo algorithm for the problem. The algorithm should run in polynomial time and return a satisfying assignment with high probability for satisfiable inputs. **[6 pts]**

Hint: View the algorithm as performing a random walk on the set $\{0, 1, \ldots, n\}$ where each state measures agreement with some satisfying assignment.

5. Let G be a 3-colorable graph. Consider the following algorithm for coloring the vertices of G with 2 colors so that no triangle of G is monochromatic. The algorithm begins with an arbitrary 2-coloring of G. While there is a monochromatic triangle in G, it chooses one such triangle, and changes the color of a randomly chosen vertex of that triangle. Derive an upper bound on the expected number of such recoloring steps before the algorithm finds a 2-coloring with the desired property. **[6 pts]**

6. **(Bonus Problem)** Let $G(X, Y, E)$ be a d-regular, connected, bipartite (multi)graph. Show that for any sets $S \subseteq X$ and $T \subseteq Y$, the number of edges connecting S and T is at least

$$\frac{\lambda_1 |S||T|}{n} - \lambda_2 \sqrt{|S||T|}.$$

As usual, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of the adjacency matrix of G. Since in a random d-regular bipartite graph, the expected number of edges from S to T is $d|S||T|/n = \lambda_1 |S||T|/n$, the statement says that larger the spectral gap, the more the graph looks like a random graph in this sense.

Hint: Consider the adjacency matrix of G premultiplied by the characteristic vector of S, and postmultiplied by the characteristic vector of T. **[6 pts]**

7. **(Bonus Problem)** Let G be an (n, d, c)-expander. Show that there exist constants $\beta, \delta > 0$ such that for any "bad" set of vertices B of cardinality at most βn, the following property holds: the probability that starting from a vertex chosen uniformly at random, a random walk of length ℓ does not visit any vertex outside of B is at most $e^{-\delta \ell}$. Exactly what properties of G are essential for your proof of this fact? **[6 pts]**