École Polytechnique Fédérale de Lausanne
Eisenbrand, Moldenhauer

December 14, 2011

Randomized Algorithms (Fall 2011)

Assignment 7

Due date: 10:00am, January 10, 2012 — 0+16 points

Registration for oral exam: write an email to carsten.moldenhauer@epfl.ch
The exams will take place on January 23 and 24. If you have any preference, please indicate this as well.

Problem 1 (3/4-Approximation for Maximum Satisfiability) — 0+2 points

Show that the combined randomized algorithm from the lecture gives a 3/4-approximation.

Problem 2 (Integrality Gap for Maximum Satisfiability) — 0+2 points

Show that the integrality gap of the following LP relaxation of Maximum Satisfiability is 3/4.

$$\begin{array}{lll} \text{maximize} & \sum_{C \in S} w_C z_C & \\ \text{subject to} & \sum_{i \in S_C^+} y_i + \sum_{i \in S_C^-} (1 - y_i) \geq z_C & C \in S \\ & 0 \leq z_C \leq 1 & C \in S \\ & 0 \leq y_i \leq 1 & i = 1, \dots, n \end{array}$$

(Hint: First, show that the integrality gap, i.e. OPT/OPT_f, is at least 3/4. Show that this is tight, i.e. 3/4 is the largest lower bound on the integrality gap, by giving an example.)

Problem 3 (Another 1/2-approximation for Maximum Satisfiability) — 0+2 points

Show that the following algorithm is a 1/2-approximation for Maximum Satisfiability. Let τ be an arbitrary truth assignment and τ' be its complement, i.e., a variable is set to 0 in τ if and only if it is set to 1 in τ'. Compute the weight of clauses satisfied by τ and τ' and output the better assignment.

Problem 4 (Derandomization) — 0+5 points

(a) Show how to derandomize the $(1 - \frac{1}{e})$-approximation algorithm from the lecture using the method of conditional expectation.

(b) Show how the combined algorithm from the lecture can be derandomized using the method of conditional expectation.

(c) Consider the following algorithm:

Algorithm Goemans-Williamson

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(1) Use the derandomized algorithm for large clauses to get a truth assignment τ_1.

(2) Use the derandomized algorithm for small clauses to get a truth assignment τ_2.

(3) Output the better of the two assignments.

Observe that this algorithm is different from the derandomized algorithm in (b). Show that the solution returned by Goemans-Williamson is always at least as good as the solution returned by the derandomized algorithm from (b).

Problem 5 (Derandomization for Maximum Satisfiability) 0+5 points

Consider the following instance for Maximum Satisfiability

$$\underbrace{(x_1 \vee x_2 \vee x_3)}_{w(C_1)=5} \wedge \underbrace{(x_1 \vee \bar{x}_2)}_{w(C_2)=4} \wedge \underbrace{(\bar{x}_1 \vee x_3)}_{w(C_3)=3} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_3)}_{w(C_4)=2} \wedge \underbrace{(x_2 \vee \bar{x}_3)}_{w(C_5)=1}.$$

with weights given by w as shown above.

Compute the truth assignment of the derandomized combined algorithm. Outline the most important steps of the computation.

Hint: $(1, 1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is an optimal solution to the LP

$$\begin{aligned}
\text{maximize}_{(\mathbf{z},\mathbf{y})} \quad & 5\,z_1 + 4\,z_2 + 3\,z_3 + 2\,z_4 + 1\,z_5 \\
\text{subject to} \quad & \\
& y_1 + y_2 + y_3 \geq z_1 \\
& y_1 + (1 - y_2) \geq z_2 \\
& (1 - y_1) + y_3 \geq z_3 \\
& (1 - y_1) + (1 - y_3) \geq z_4 \\
& y_2 + (1 - y_3) \geq z_5 \\
& 0 \leq z_i \leq 1 \qquad (1 \leq i \leq 5) \\
& 0 \leq y_j \leq 1 \qquad (1 \leq j \leq 3).
\end{aligned}$$