École Polytechnique Fédérale de Lausanne
Eisenbrand, Moldenhauer

December 7, 2011

Randomized Algorithms (Fall 2011)

Assignment 6

Due date: 10:00am, December 20, 2011 | 25+16 points

Problem 1 (Independences) 5 points

Prove or find a counter example.

(a) $A_1, \dots, A_n$ are mutually independent if and only if $\bar{A}_1, \dots, \bar{A}_n$ are mutually independent.

(b) A is mutually independent from $B_1, \dots, B_n$ if and only if A is mutually independent from $\bar{B}_1, B_2, \dots, B_n$.

(c) Let $A_1, \dots, A_n$ be mutually independent and $\Pr(A_i) < 1$ for all $1 \le i \le n$. Then $\Pr(\bar{A}_1 \cap \dots \cap \bar{A}_n) = \prod_{i=1}^n (1 - \Pr(A_i)) > 0$. (without using the Lovasz Local Lemma).

Problem 2 (Symmetric Local Lemma) 3 points

By using the Lovasz Local Lemma in its general form, prove the following symmetric version.
Let $A_1, \dots, A_n$ be events and $D = (V, E)$ be their dependence digraph. Assume further that $\deg(A_i) \le d$, $\Pr(A_i) \le p$ and $ep(d+1) \le 1$. Then

$$\Pr(\bigcap_{i=1}^{n} \bar{A}_i) > 0 \; .$$

Problem 3 (Routing dogs) 5 points

There are n dogs living in the city of Dogtown. Every morning, the owner of dog i can choose from a set of m paths P_i through town to take his dog for a walk. For different dogs i and j at most k of the paths in P_i intersect with paths in P_j.
If the owners of two dogs chose intersecting paths, the dogs will mark their territory at the intersection point. To minimize the amount of damage by feces, the city council wonders if if is possible to select a path for each dog such that no two dog paths intersect.
Find the maximum value of k that you can come up with such that this is always possible.

Problem 4 (Constructive Local Lemma) 7 points

Consider the algorithm presented in the lecture to solve k-SAT with limited interaction (for each clause there is at most 2^{k-C} other clauses that share a variable with it). We showed that the algorithm's execution can be used to encode a random bit string and that after sufficiently many iterations M, this encoding is smaller than the number of random bits. This gives the argument that the algorithm cannot run infinitely long for any input (of random bits).

(a) Derive the best upper bound on M that you can come up with such that the above argument holds.

(b) Show which number M of iterations is sufficient such that the probability that the algorithm runs for more than M iterations is at most 1/2.
(Hint: Let $b(M)$ be the number of bits that the algorithm considers in M iterations. Assume that on half of the random strings of size $b(M)$ the algorithm would continue computation beyond M iterations. This means that at least half of the strings of size $b(M)$ can be encoded by the run of the algorithm. Derive a contradiction by choosing M appropriately.)

Problem 5 (Independent Set) 5 points

Consider a graph $G = (V, E)$ on n vertices and the following method of generating an independent set (*i.e.*, a set I such that $\forall i, j \in I : \{i, j\} \notin E$). Given a permutation σ of the vertices, define a subset $S(\sigma)$ of the vertices as follows: for each vertex i, $i \in S(\sigma)$ if and only if no neighbor j of i precedes i in the permutation σ.

(a) Show that each $S(\sigma)$ is an independent set in G.

(b) Suggest a randomized algorithm to produce σ such that the expected cardinality of $S(\sigma)$ is $\sum_{i=1}^{n} \frac{1}{d_i+1}$, where d_i denotes the degree of vertex i.

(c) Prove that G has an independent set of size at least $\sum_{i=1}^{n} 1/(d_i + 1)$.

Problem 6 (Token Game) 0+5 points

Consider the following two-player game. The game begins with k tokens placed at the number 0 on the integer number line spanning $[0, n]$. Each round, one player, called the chooser, selects two disjoint and nonempty sets of tokens A and B. (The sets A and B need not cover all the remaining tokens). The second player, called the remover, takes all the tokens from one of the sets of the board. The tokens from the other set all move up one space on the number line from their current position. The chooser wins if any token ever reaches n. The remover wins if the chooser finishes with one token that has not reached n.

(a) Give a winning strategy for the chooser when $k \geq 2^n$.

(b) Give a deterministic winning strategy for the remover when $k < 2^n$.
(Hint: prove that the obvious strategy is a winning strategy using the following potential function: the sum of $2^{\mathrm{pos}(i)}$ for all remaining tokens where $\mathrm{pos}(i)$ denotes the position of remaining token i).

(c) Use the probabilistic method to show that there must exist a winning strategy for the remover when $k < 2^n$.

(d) Explain how to use the method of conditional expectation to derandomize the winning strategy for the remover when $k < 2^n$.

Problem 7 (Monochromatic subgraphs) 0+4 points

(a) Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.

(b) Show how to construct such a coloring deterministically in polynomial time.
(Hint: use the method of conditional expectation)

Problem 8 (Coloring) 0+3 points

Let $G = (V, E)$ be an undirected graph and suppose each $v \in V$ is associated with a set $S(v)$ of $6r$ colors, where $r \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most r neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$ such that, for any edge $(u, v) \in E$, the colors assigned to u and v are different.
(Hint: Let $A_{u,v,c}$ be the event that u and v are both colored with c and apply the Local Lemma).

Problem 9 (Independent Set) 0+4 points

Let $G = (V, E)$ be a cycle of length $4n$ and let $V = V_1 \cup \cdots \cup V_n$ be a partition of its $4n$ vertices into n pairwise disjoint subsets, each of cardinality 4. Prove or disprove: There exists an independent set of G containing precisely one vertex from each V_i.