Weighted Digraphs and Tropical Cones

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Scheduling with AND/OR precedence constraints

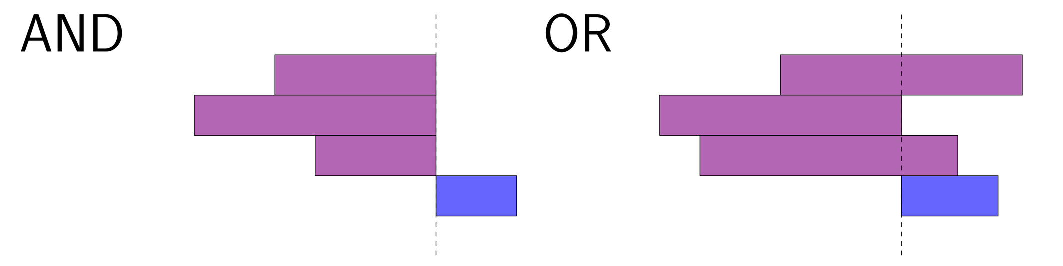
Network Given a set of jobs V (AND-nodes) and waiting conditions W (OR-nodes). They form a directed bipartite graph $\mathcal{B}$ with the arc set A.

Processing times/ time lags integral weights $-M < d_{pq} < M$ for $(p,q) \in A$.

Starting condition $S_j \geq 0$ for all $j \in V$.

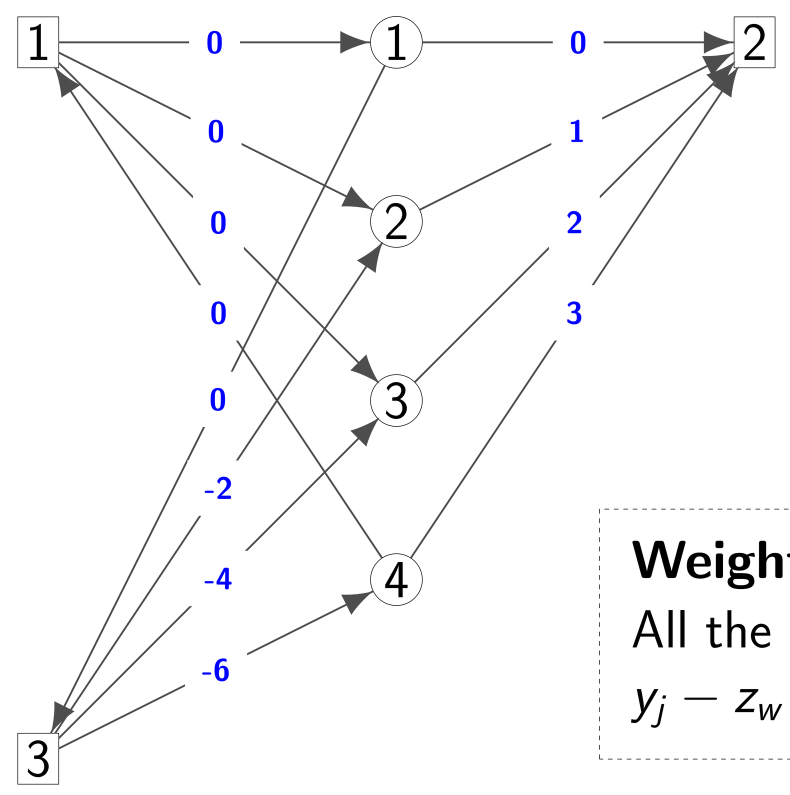
Problem formulation

$$S_j \geq \max_{(w,j)\in A} (S_w + d_{wj}) \qquad \text{for } j \in V$$
$$S_w \geq \min_{(j,w)\in A} (S_j + d_{jw}) \qquad \text{for } w \in W$$



Theorem (J.,L. 2016+)
The orthogonal projection of the set of feasible schedules onto the coordinates in V is the tropical polyhedron given by

$$\min_{(w,j)\in A} (S_j - d_{wj}) \geq \min_{(j,w)\in A} (S_j + d_{jw}) \qquad \forall w \in W \ .$$



$$\text{minimize } (\infty, 0, 0) \odot \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned} 0 &\leq \min(0 + x_2, 0 + x_3) \\ \min(0, x_3 - 2) &\leq x_2 - 1 \\ \min(0, x_3 - 4) &\leq x_2 - 2 \\ x_3 - 6 &\leq \min(0, x_2 - 3) \end{aligned}$$

Weighted digraph polyhedron for $\mathcal{B}$
All the points $(y,z) \in \mathbb{R}^{V \sqcup W}$ with
$y_j - z_w \leq d_{jw}$ and $y_j - z_w \leq d_{wj}$ for all arcs.

Tropical Polyhedra

Tropical numbers $\mathbb{T} = \mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$ each with ∞

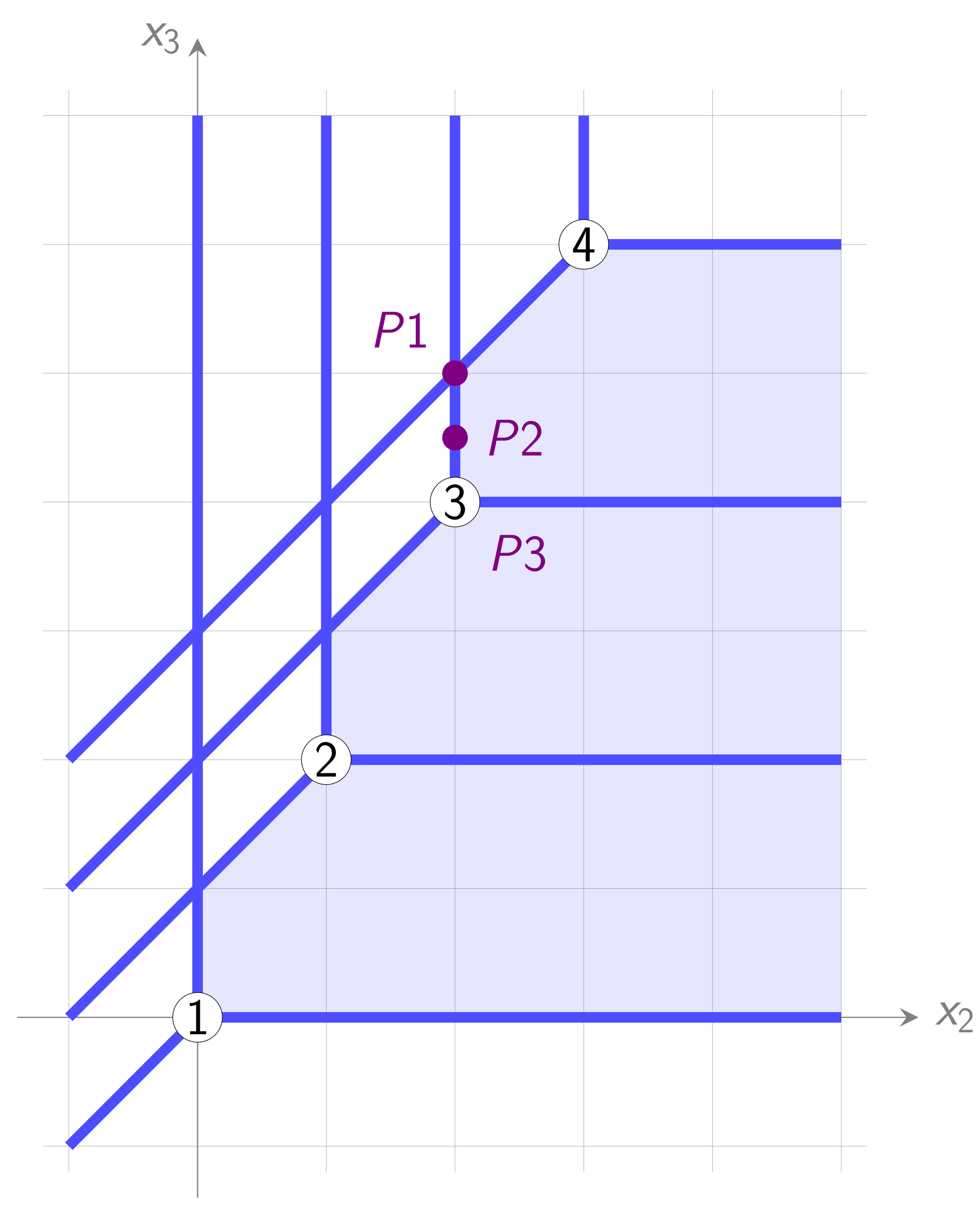
Tropical operations $\oplus = \min$ and $\odot = +$
Remark: There is no additive inverse.

Inequality systems Systems of the form $A \odot x \leq B \odot x$ define tropical polyhedra where
$A \odot x = (\bigoplus_{k=1}^{d} a_{ik} \odot x_k)_i$

Theorem (Möhring, Skutella, Stork)
The following problems are polynomial time equivalent and belong to NP ∩ co-NP
- Finding a minimal schedule in a network with and/or-precedence constraints.
- Checking the feasibility of a tropical polyhedron.

- No polynomial time algorithm known
- Feasibility is equivalent to tropical linear programming, bisection or homogenization approach

Theorem (J., L. 2015)
The set of feasible schedules is described by those bipartite subgraphs of $\mathcal{B}$ in which every node in W has at least one in-going arc. These subgraphs are directed covector graphs.



covector graphs describe the local geometry and position

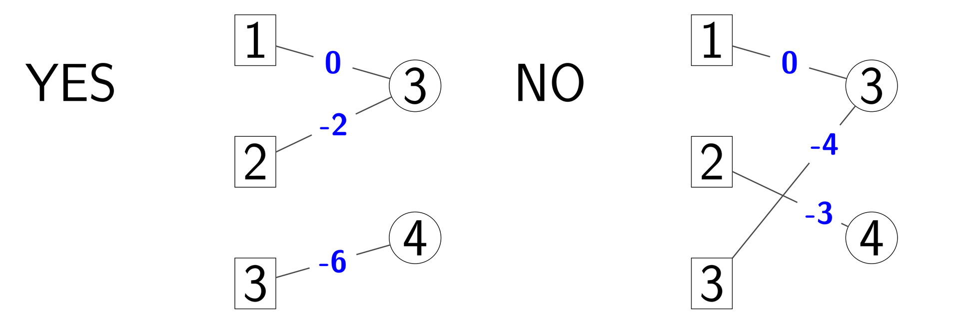
Tropical covector graphs

A bipartite graph G on $V \sqcup W$ is a *covector graph* for a weight matrix $D \in (\mathbb{R} \cup \{\infty\})^{V \times W}$ if and only if the following are satisfied

Minimality: for every pair of subsets $P \subseteq V$ and $Q \subseteq W$ with $|P| = |Q|$, every perfect matching of G restricted to $P \sqcup Q$ is a minimal matching of the complete bipartite graph $P \times Q$ with the weights given by the corresponding submatrix of D;

Completeness: if there are more minimal perfect matchings in $P \times Q$ then each of them is contained in G.

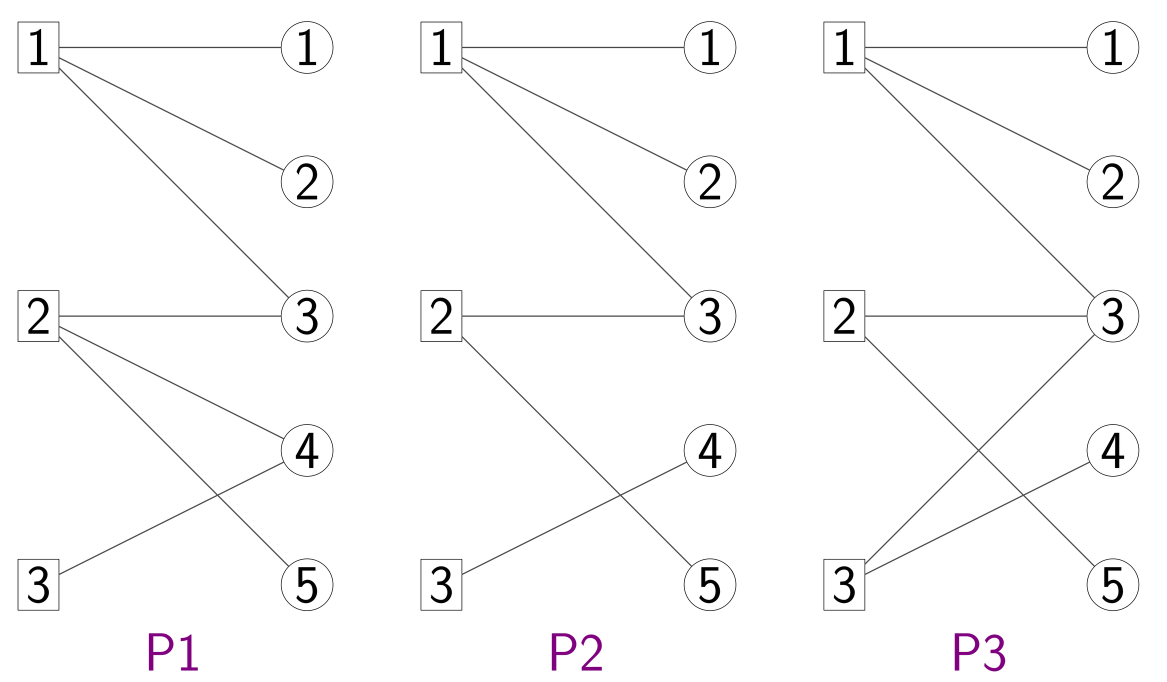
- Tropical oriented matroid
- Similar to graphs in the hungarian method
- Shortest path in bipartite graph

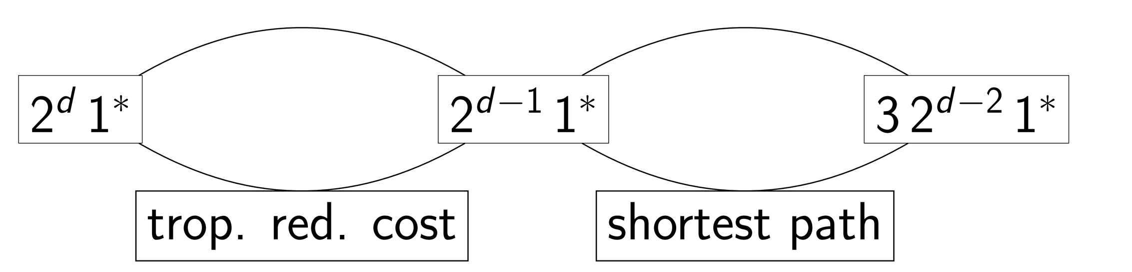


Tropical linear programming - a graph algorithm

Theorem (Allamigeon, Benchimol, Gaubert, J.)
For every generic instance of the tropical simplex method there is a classical analogue. Both are polynomial-time equivalent.

- New approach with promising complexity
- Relation between classical simplex method for arbitrary polyhedra and shortest path algorithms
- Genericity no restriction via symbolic perturbation





References

- Möhring, Rolf H. and Skutella, Martin and Stork, Frederik *Scheduling with AND/OR precedence constraints*, SIAM J. Comput. 33 (2004) 2
- Allamigeon, Xavier and Benchimol, Pascal and Gaubert, Stéphane and Joswig, Michael *Combinatorial simplex algorithms can solve mean payoff games*, SIAM J. Opt. 24 (2014) 4
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