Abstract tropical linear programming

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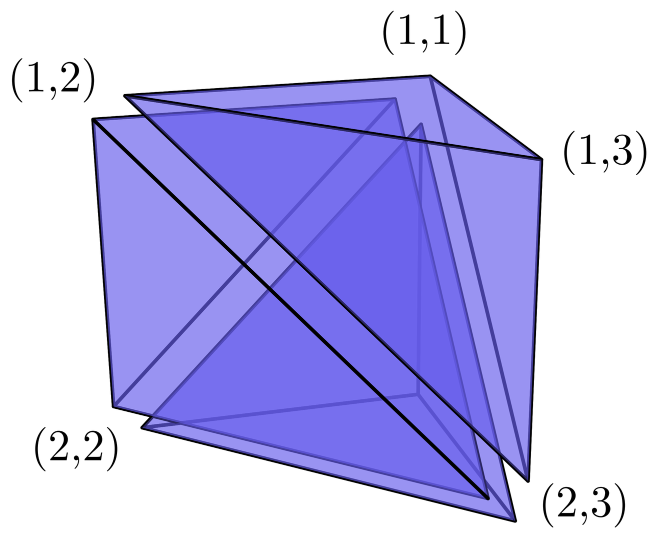
Signed tropical matroids

Definition

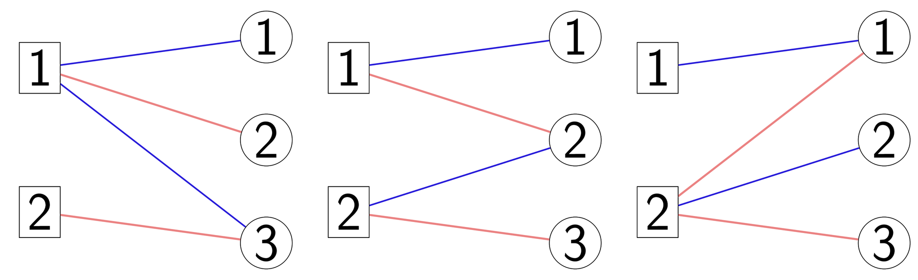
- Set of bipartite graphs $\mathcal{T}$ corresponding to the cells in a subdivision of a subpolytope $\Delta_{d-1} \times \Delta_{n-1}$
- Sign matrix $\Sigma \in \{+, -, \bullet\}^{d \times n}$

Background:

- Description of subdivision of $\Delta_{d-1} \times \Delta_{n-1}$ as *Tropical Oriented Matroids* (Ardila, Develin 2009; Oh, Yoo 2011; Horn 2016)
- Regular subdivisions in bijection with tropical point configurations (Develin, Sturmfels 2004; Fink, Rincón 2015; Joswig, Loho 2016)



$$\begin{pmatrix} + & - \\ - & + \\ + & - \end{pmatrix}$$



Simplex Method (Dantzig '63, Bland '77,...)

Given $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$. Consider

$$A \cdot x \leq b \ .$$

Problem: Find a feasible point $y \in \mathbb{R}^d$ fulfilling the system.

1: $I \leftarrow$ appropriate d-elem. subset of $[n]$ (basis of rows)
2: $y \leftarrow$ solution of $A_I \cdot x = b_I$
3: **while** y does not fulfill the system and no certificate for infeasibility found **do**
4: $\quad f \leftarrow$ particular element of $[n] \setminus I$
5: $\quad e \leftarrow$ particular element of I
6: $\quad I \leftarrow I \cup \{f\} \setminus \{e\}$ (again a basis)
7: $\quad y \leftarrow$ solution of $A_I \cdot x = b_I$
8: **end while**
9: **return** y

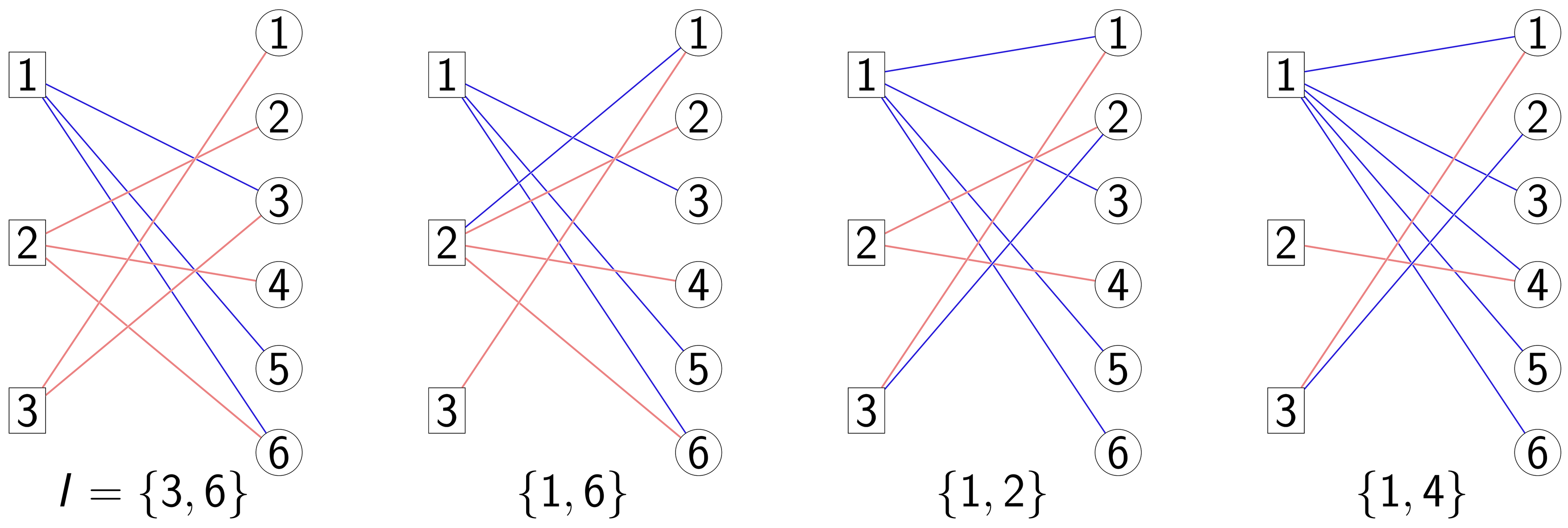
Abstract Tropical Feasibility Algorithm (Loho 2017)

Problem: Find feasible Cramer covector.

1: $I \leftarrow$ appropriate d-elem. subset of $[n]$
2: **while** There is $j \in [n]$ only incident with negative edges in Cramer covector of I (and it is not totally infeasible) **do**
3: $\quad k \leftarrow$ node in $[n]$ incident with same node in $[d+1]$ via negative edge
4: $\quad I \leftarrow I \setminus \{k\} \cup \{j\}$
5: **end while**
6: **return** I

Theorem (L 2017)

The algorithm works in a not-necessarily regular triangulation of $\Delta_d \times \Delta_{n-1}$ and returns a feasible Cramer covector or a witness that there is no feasible Cramer covector.



Basic Covectors

Fix $I \subset [n]$ of size d.

Classical (Cramer 1750)

$$(A_I | - b_I) \cdot x = 0 \ .$$

Solution vector given by $d \times d$-subdeterminants of $(A_I | - b_I)$.

Tropical ('Max Plus' '90, RGST 2005, AGG 2014)

min attained twice per row in $M_I \odot_{\min} x = (\min_{j \in [d+1]}(m_{ij} + x_j))_{i \in I}$

- Solution given by tropical $d \times d$-subdet. of $M \in (\mathbb{R} \cup \infty)^{n \times (d+1)}$.
- Tropical determinants are minimal $d \times d$ matchings.

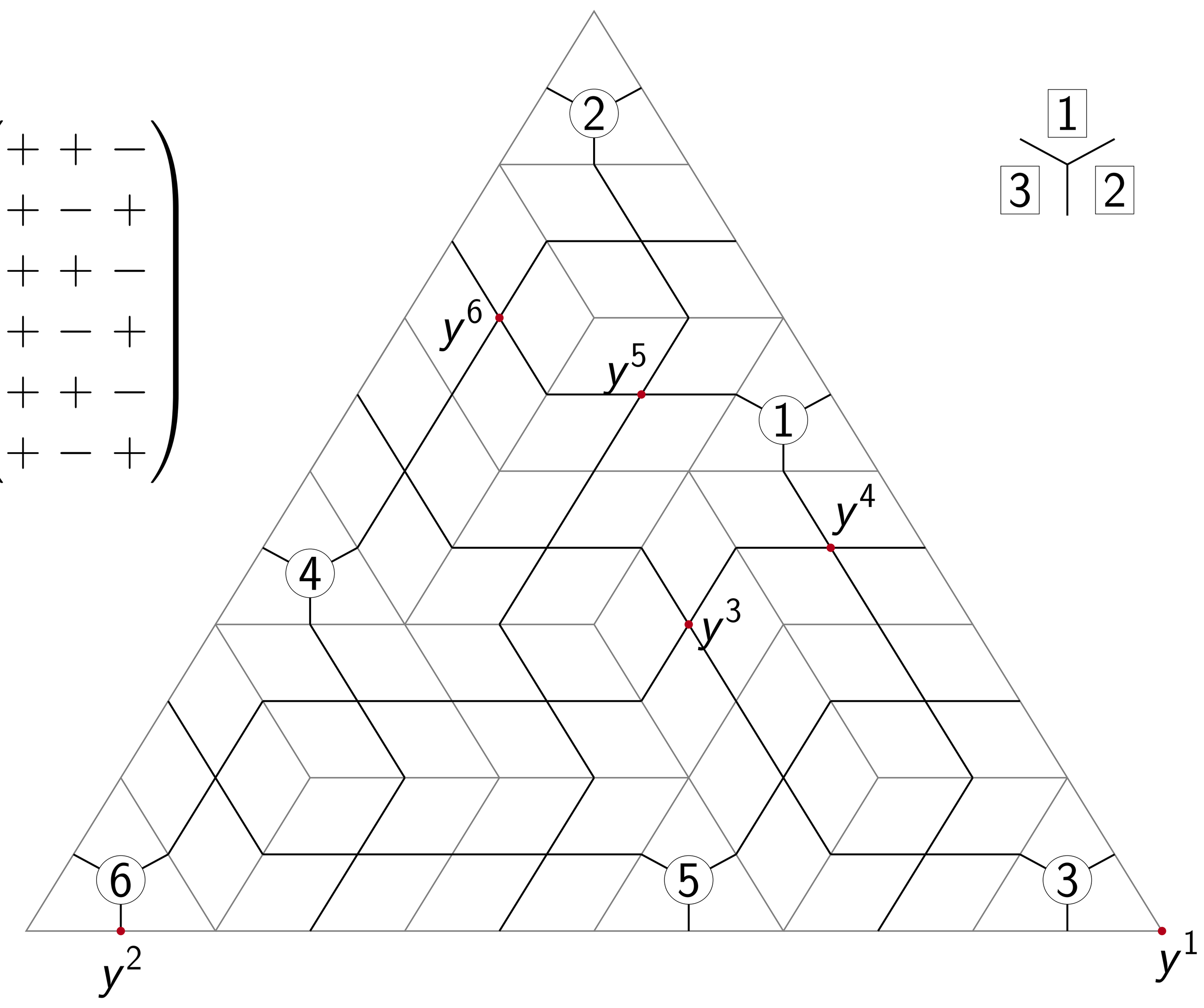
Abstract Tropical (L 2017)

Cramer covector of I: Bipartite tree on $[d+1] \sqcup [n]$ such that d nodes in $I \subset [n]$ have degree 2 (and the others are of degree 1).

- Generalization of 'min attained twice', special covector graphs.

TLP in a Non-Regular Triangulation of $\Delta_2 \times \Delta_5$

$$\begin{pmatrix} + & + & - \\ + & - & + \\ + & + & - \\ + & - & + \\ + & + & - \\ + & - & + \end{pmatrix}$$



Combinatorial Progress measure

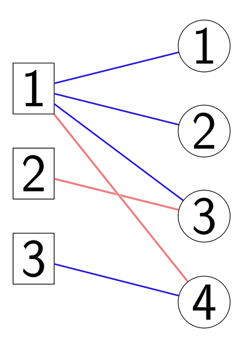
- Affine coordinates of a point with given covector are the weights of the paths from fixed node to coordinate nodes.
- Purely combinatorial criterion on the paths using the concept of 'odd' and 'even' edges

Covectors from Tropical Matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \\ 0 & \infty & -6 \end{pmatrix}, \begin{pmatrix} + & - & - \\ + & - & + \\ + & - & + \\ - & \bullet & + \end{pmatrix}$$

Inequality system

$$\begin{aligned} x_1 &\leq \min(x_2, x_3) \\ \min(x_1, x_3 - 2) &\leq x_2 - 1 \\ \min(x_1, x_3 - 4) &\leq x_2 - 2 \\ x_3 - 6 &\leq x_1 \ . \end{aligned}$$



$(-8, -6, -2)$

Runtime Analysis and the Secondary Fan of $\Delta_{d-1} \times \Delta_{n-1}$

Theorem (L 2017)

The algorithm takes $\mathcal{O}(d\omega)$ steps for a tropical linear inequality system given by a matrix $M \in (\mathbb{Z} \cup \{\infty\})^{n \times (d+1)}$.
Here, ω is the maximal finite entry of a fixed non-negative integer matrix which induces the same regular subdivision of $\Delta_d \times \Delta_{n-1}$ as M.

References

Abstract tropical linear programming, Georg Loho, 2017, arXiv:1612.01890