Polyhedra

In this chapter we give definitions and fundamental facts about polyhedra. An excellent reference for this topic is the book by Schrijver [1]. A *polyhedron* P is a set of vectors of the form

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\},$$

for some matrix $A \in \mathbb{R}^{m\times n}$ and some vector $b \in \mathbb{R}^m$. We write $P = P(A, b)$. The polyhedron is *rational* if both A and b can be chosen to be rational.

We review some notation. Let $X \subseteq \mathbb{R}^n$ be a set of n-dimensional vectors. The *linear hull*, *affine hull* and *convex hull* of X are defined as follows:

$$\begin{aligned} \text{lin.hull}(A) = &\{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geq 0, \\ &x_1, \ldots, x_t \in X,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\} \end{aligned} \tag{1}$$

$$\begin{aligned} \text{aff.hull}(A) = &\Bigg\{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geq 1, \\ &x_1, \ldots, x_t \in X,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\Bigg\} \end{aligned} \tag{2}$$

$$\begin{aligned} \text{conv}(A) = &\Bigg\{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geq 1, \\ &x_1, \ldots, x_t \in X,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}_{\geq 0}\Bigg\} \end{aligned} \tag{3}$$

For $x_0 \in \mathbb{R}^n$ and $X, Y \subseteq \mathbb{R}^n$, we denote

$$\begin{aligned} X + Y &= \{x + y \mid x \in X,\ y \in Y\}, \\ x_0 + X &= \{x_0 + x \mid x \in X\}. \end{aligned}$$

Proposition 1. *Let $X \subseteq \mathbb{R}^n$ and $x_0 \in X$. Then*

$$\text{aff.hull}(X) = x_0 + \text{lin.hull}(X - x_0).$$

Proof. One has $x \in \text{aff.hull}(X)$ if and only if $x = \lambda_0 x_0 + \lambda_1 x_1 + \cdots + \lambda_t x_t$ for some $x_1, \ldots, x_t \in X$ ($t \geq 0$) such that $\sum_{i=0}^{t} \lambda_i = 1$. Then

$$\begin{aligned} x &= x_0 + \lambda_0(x_0 - x_0) + \lambda_1(x_1 - x_0) + \cdots + \lambda_t(x_t - x_0) \\ &= x_0 + \lambda_1(x_1 - x_0) + \cdots + \lambda_t(x_t - x_0). \end{aligned}$$

This shows the claim. □

A finite set $V \subseteq \mathbb{R}^n$ is called *affinely independent* if for each $v \in V$, one has $v \notin \text{aff.hull}(V \setminus \{v\})$. This is equivalent to $(V - v) \setminus \{0\}$ being linearly independent for each $v \in V$. The *dimension* of V is the size of the largest subset of V which is affinely independent.

An inequality $a^\mathsf{T} x \leq \beta$ is called an *implicit equality* of $Ax \leq b$ if each $x^* \in P(A, b)$ satisfies $a^\mathsf{T} x^* = \beta$. We denote the subsystem consisting of implicit equalities of $Ax \leq b$ by $A^= x \leq b^=$ and the subsystem consisting of the other inequalities by $A^\leq x \leq b^\leq$. An inequality is *redundant* if its removal from $Ax \leq b$ does not change the set of feasible solution of $Ax \leq b$.

Lemma 2. *There exists an $x \in P(A, b)$ with $A^\leq x < b^\leq$.*

Proof. Suppose that the inequalities in $A^\leq x \leq b^\leq$ are

$$a_1^\mathsf{T} x \leq \beta_1, \ldots, a_k^\mathsf{T} x \leq \beta_k.$$

For each $1 \leq i \leq k$ there exists an $x_i \in P$ with $a_i^\mathsf{T} x_i < \beta_i$. Then the point

$$x = \frac{1}{k}(x_1 + \cdots + x_k)$$

is a point of $P(A, b)$ satisfying $A^\leq x < b^\leq$. □

Lemma 3. *Let $Ax \leq b$ be a system of linear inequalities. One has*

$$\text{aff.hull}(P(A, b)) = \{x \in \mathbb{R}^n \mid A^= x = b^=\} = \{x \in \mathbb{R}^n \mid A^= x \leq b^=\}.$$

Proof. Let $x_1, \ldots, x_t \in P(A, b)$ and suppose that $a^\mathsf{T} x \leq \beta$ is an implicit equality. Then since $a^\mathsf{T} x_i = \beta$, one has

$$a^\mathsf{T} \left(\sum_{j=1}^{t} \lambda_i x_i \right) = \beta.$$

Therefore the inclusions $\subseteq$ follow.

Suppose now that x_0 satisfies $A^= x \leq b^=$. Let $x_1 \in P(A, b)$ with $A^\leq x_1 < b^\leq$. If $x_0 = x_1$ then $x_0 \in P(A, b) \subseteq \text{aff.hull}(P(A, b))$. Otherwise the line segment between x_0 and x_1 contains more than one point in P and thus $x_1 \in \text{aff.hull}(P)$. □

Decomposition theorem for polyhedra

A nonempty set $C \subseteq \mathbb{R}^n$ is a *cone* if $\lambda x + \mu y \in C$ for each $x, y \in C$ and $\lambda, \mu \in \mathbb{R}_{\geq 0}$. A cone C is *polyhedral* if

$$C = \{x \in \mathbb{R}^n \mid Ax \leq 0\}.$$

A cone *generated by* vectors $x_1, \ldots, x_m \in \mathbb{R}^n$ is a set of the form

$$C = \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid \lambda_i \in \mathbb{R}_{\geq 0}, \ i = 1, \ldots, m \right\}.$$

A point

$$x = \sum_{i=1}^{m} \lambda_i x_i, \qquad \lambda_i \in \mathbb{R}_{\geq 0},\, i = 1, \ldots, m$$

is called a *conic combination* of $x_1, \ldots, x_m$. The set of conic combinations of X is denoted by $\operatorname{cone}(X)$.

Theorem 4 (Farkas–Minkowsi–Weyl theorem). *A convex cone is polyhedral if and only if it is finitely generated.*

Proof. Suppose that $a_1, \ldots, a_m$ span $\mathbb{R}^n$ and consider the cone

$$C = \left\{ \sum_{i=1}^{m} \lambda_i a_i \mid \lambda_i \geq 0,\, i = 1, \ldots, m \right\}.$$

Suppose that $b \notin C$ holds. Then the linear program

$$\min\{0^{\mathsf{T}} x \mid (a_1, \ldots, a_m) x = b,\, x \geq 0\}$$

does not have a feasible solution. Its dual is

$$\max\{b^{\mathsf{T}} y \mid A^{\mathsf{T}} y \leq 0\},$$

with $A = (a_1, \ldots, a_m)$. The dual program is feasible and, by the duality theorem, unbounded.

This shows that there exists a $y^* \in \mathbb{R}^n$ with

$$\begin{aligned} b^{\mathsf{T}} y^* &> 0, \\ a_i^{\mathsf{T}} y^* &\leq 0 \qquad \text{for each } i = 1, \ldots, m. \end{aligned}$$

Suppose that the columns of A which correspond to inequalities in $A^{\mathsf{T}} y \leq 0$ that are satisfied by y^* with equality have rank strictly smaller than $n - 1$. Denote these columns by $a_{i_1}, \ldots, a_{i_k}$. Then there exists a nonzero vector v which is orthogonal to each of these columns and to b, i.e.,

$$\begin{aligned} a_{i_j}^{\mathsf{T}} v &= 0 \qquad \text{for each } j = 1, \ldots, k \\ b^{\mathsf{T}} v &= 0. \end{aligned}$$

There also exists a column a^* of A which is not in the set $\{a_{i_1}, \ldots, a_{i_k}\}$ such that $(a^*)^{\mathsf{T}} v > 0$, since the columns of A span $\mathbb{R}^n$. Therefore there exists an $\varepsilon > 0$ such that

(i) $A^{\mathsf{T}}(y^* + \varepsilon \cdot v) \leq 0$;

(ii) the subspace generated by the columns of A which correspond to inequalities of $A^{\mathsf{T}} y \leq 0$ which are satisfied by $y^* + \varepsilon \cdot v$ with equality strictly contains $\operatorname{lin.hull}(a_{i_1}, \ldots, a_{i_k})$.

Notice that we have $b^{\mathsf{T}}y^* = b^{\mathsf{T}}(y^* + \epsilon \cdot v) > 0$.

Continuing this way, we obtain a solution of the form $y^* + u$ of $A^{\mathsf{T}}y \leq 0$ such that one has $n-1$ linearly independent columns of A whose corresponding inequality in $A^{\mathsf{T}}y \leq 0$ are satisfied with equality. Thus we see that each b which does not belong to C can be separated from C with an inequality of the form $c^{\mathsf{T}}y \leq 0$ which is uniquely (up to scaling) defined by $n-1$ linearly independent vectors from the set $a_1, \ldots, a_m$. This shows that C is polyhedral.

Suppose now that $a_1, \ldots, a_m$ do not span $\mathbb{R}^n$. Then there exist linearly independent vectors $d_1, \ldots, d_k$ such that each d_i is orthogonal to each of the $a_1, \ldots, a_m$ and $a_1, \ldots, a_m, d_1, \ldots, d_k$ span $\mathbb{R}^n$. The cone generated by $a_1, \ldots, a_m, d_1, \ldots, d_k$ is polyhedral and thus of the form $Ax \leq 0$ with some matrix $A \in \mathbb{R}^{m \times n}$. Suppose that

$$\text{lin.hull}(a_1, \ldots, a_m) = \{x \in \mathbb{R}^n \mid Ux = 0\}.$$

Now

$$C = \{x \in \mathbb{R}^n \mid Ax \leq 0, Ux = 0\}$$

and C is polyhedral.

Now suppose that

$$C = \{x \in \mathbb{R}^n \mid a_1^{\mathsf{T}}x \leq 0, \ldots, a_m^{\mathsf{T}}x \leq 0\}.$$

The cone

$$\text{cone}(a_1, \ldots, a_m) = \left\{ \sum_{i=1}^m \lambda_i a_i \mid \lambda_i \geq 0, i = 1, \ldots, m \right\}$$

is polyhedral and thus of the form

$$\text{cone}(a_1, \ldots, a_m) = \{x \in \mathbb{R}^n \mid b_1^{\mathsf{T}}x \leq 0, \ldots, b_k^{\mathsf{T}}x \leq 0\}.$$

Clearly $\text{cone}(b_1, \ldots, b_k) \subseteq C$, since $b_i^{\mathsf{T}}a_j \leq 0$. Suppose that $y \in C \setminus \text{cone}(b_1, \ldots, b_k)$. Then, since $\text{cone}(b_1, \ldots, b_k)$ is polyhedral, there exists a $w \in \mathbb{R}^n$ with

$$\begin{aligned} w^{\mathsf{T}}y &> 0 \\ w^{\mathsf{T}}b_i &\leq 0 \qquad \text{for each } i = 1, \ldots, k. \end{aligned}$$

From the latter we conclude that $w \in \text{cone}(a_1, \ldots, a_m)$. Since $y \in C$ and $w \in \text{cone}(a_1, \ldots, a_m)$, we conclude that $w^{\mathsf{T}}y \leq 0$. □

A set of vectors $Q = \text{conv}(X)$, where $X \subseteq \mathbb{R}^n$ is finite, is called a *polytope*.

Theorem 5 (Decomposition theorem for polyhedra)**.** *A set $P \subseteq \mathbb{R}^n$ is a polyhedron if and only if $P = Q + C$ for some polytope Q and a polyhedral cone C.*

Proof. Suppose $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is a polyhedron. The polyhedral cone

$$\{ \left(\begin{smallmatrix} x \\ \lambda \end{smallmatrix}\right) \mid x \in \mathbb{R}^n, \lambda \in \mathbb{R}_{\geq 0}; Ax - \lambda b \leq 0\} \tag{4}$$

is generated by finitely many vectors $\left(\begin{smallmatrix} x_i \\ \lambda_i \end{smallmatrix}\right)$, $i = 1, \ldots, m$. By scaling with a positive number we may assume that each $\lambda_i \in \{0, 1\}$. Let Q be the convex hull of the x_i

with $\lambda_i = 1$ and let C be the cone generated by the x_i with $\lambda_i = 0$. A point $x \in \mathbb{R}^n$ is in P if and only if $\binom{x}{1}$ belongs to (4) and thus if and only if

$$\binom{x}{1} \in \text{cone}\left\{\binom{x_1}{\lambda_1}, \ldots, \binom{x_m}{\lambda_m}\right\}.$$

Therefore $P = Q + C$.

Suppose now that $P = Q + C$ for some polytope Q and a polyhedral cone C with $Q = \text{conv}(x_1, \ldots, x_m)$ and $C = \text{cone}(y_1, \ldots, y_t)$. A vector x_0 is in P if and only if

$$\binom{x_0}{1} \in \text{cone}\left\{\binom{x_1}{1}, \ldots, \binom{x_m}{1}, \binom{y_1}{0}, \ldots, \binom{y_t}{0}\right\} \tag{5}$$

By Theorem 4, the cone in (5) is equal to

$$\left\{\binom{x}{\lambda} \mid Ax - \lambda b \leq 0\right\} \tag{6}$$

for some matrix A and vector b. Thus $x_0 \in P$ if and only if $Ax_0 \leq b$ and thus P is a polyhedron. $\square$

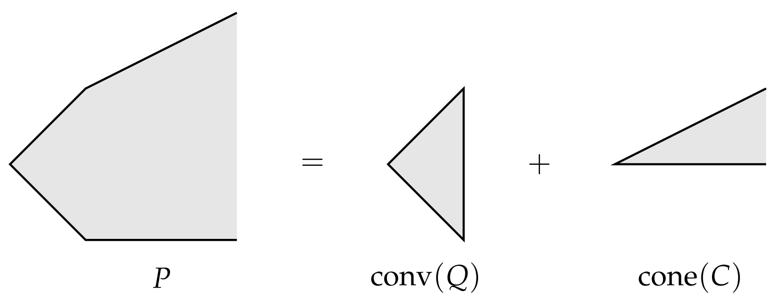

Figure 1: A polyhedron and its decomposition into Q and C

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. The *characteristic cone* of P is

$$\text{char.cone}(P) = \{y \mid y + x \in P \text{ for all } x \in P\} = \{y \mid Ay \leq 0\}.$$

One has

(i) $y \in \text{char.cone}(P)$ if and only if there exists an $x \in P$ such that $x + \lambda y \in P$ for all $\lambda \geq 0$;

(ii) $P + \text{char.cone}(P) = P$;

(iii) P is bounded if and only if $\text{char.cone}(P) = \{0\}$;

(iv) if the decomposition of P is $P = Q + C$, then $C = \text{char.cone}(P)$.

The *linearity space* of P is defined as $\text{char.cone}(P) \cap (-\text{char.cone}(P))$. A polyhedron is *pointed*, if its lineality space is $\{0\}$.

Exercise 1. Each nonempty polyhedron $P \subseteq \mathbb{R}^n$ can be represented as $P = L + Q$, where $L \subseteq \mathbb{R}^n$ is a linear space and $Q \subseteq \mathbb{R}^n$ is a pointed polyhedron.

Faces

An inequality $c^{\mathsf{T}}x \leq \delta$ is called *valid* for P if each $x \in P$ satisfies $c^{\mathsf{T}}x \leq \delta$. If in addition $\{x \mid c^{\mathsf{T}}x = \delta\} \cap P \neq \emptyset$, then $c^{\mathsf{T}}x \leq \delta$ is a *supporting inequality* and $c^{\mathsf{T}}x = \delta$ is a supporting hyperplane.

A set $F \subseteq \mathbb{R}^n$ is called a *face* of P if there exists a valid inequality $c^{\mathsf{T}}x \leq \delta$ for P with $F = P \cap \{x \mid c^{\mathsf{T}}x = \delta\}$.

Lemma 6. *Let $F \neq \emptyset$ be a nonempty face of P, then $F = \{x \in P \mid A'x = b'\}$ for a subsystem $A'x \leq b'$ of $Ax \leq b$.*

Proof. Suppose that $F = \{x \in P \mid A'x = b'\}$. Consider the vector $c^{\mathsf{T}} = \mathbb{1}^{\mathsf{T}}A'$ and $\delta = \mathbb{1}^{\mathsf{T}}b'$. The inequality $c^{\mathsf{T}}x \leq \delta$ is valid for P. It is satisfied with equality by each $x \in F$. If $x' \in P \setminus F$, then there exists an inequality $a^{\mathsf{T}}x \leq \beta$ in $A'x \leq b'$ such that $a^{\mathsf{T}}x' < \beta$ and consequently $c^{\mathsf{T}}x' < \delta$.

On the other hand, if $c^{\mathsf{T}}x \leq \delta$ defines the face F, then $c^{\mathsf{T}} = \lambda^{\mathsf{T}}A$ and $\delta = \lambda^{\mathsf{T}}b$ with $\lambda \in \mathbb{R}^m_{\geq 0}$. Let $A'x \leq b'$ be the subsystem of $Ax \leq b$ which corresponds to strictly positive entries in $Ax \leq b$. One has $F = \{x \in P \mid A'x = b'\}$. ☐

Exercise 2. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ and consider the polyhedron $P = P(A, b)$. Show that $\dim(P) = n - \operatorname{rank}(A^=)$

A *facet* of P is an inclusion-wise maximal face F of P with $F \neq P$. An inequality $a^{\mathsf{T}}x \leq \beta$ of $Ax \leq b$ is called *redundant* if $P(A, b) = P(A', b')$, where $A'x \leq b'$ is the system stemming from $Ax \leq b$ by deleting $a^{\mathsf{T}}x \leq \beta$. A system $Ax \leq b$ is *irredundant* if $Ax \leq b$ does not contain a redundant inequality.

Lemma 7. *Let $Ax \leq b$ be an irredundant system. Then F is a facet of P if and only if F has the form*

$$F = \{x \in P \mid a^{\mathsf{T}}x = \beta\}$$

for an inequality $a^{\mathsf{T}}x \leq \beta$ of $A^{\leq}x \leq b^{\leq}$.

Proof. Let F be a facet of P. Then $F = \{x \in P \mid c^{\mathsf{T}}x = \delta\}$ for a valid inequality $c^{\mathsf{T}}x \leq \delta$ of P. There exists a $\lambda \in \mathbb{R}^m_{\geq 0}$ with $c^{\mathsf{T}} = \lambda^{\mathsf{T}}A$ and $\delta = \lambda^{\mathsf{T}}b$. There exists an inequality $a^{\mathsf{T}}x \leq \beta$ of $A^{\leq}x \leq b^{\leq}$ whose corresponding entry in λ is strictly positive. Clearly $F \subseteq \{x \in P \mid a^{\mathsf{T}}x = \beta\} \subset P$. Since F is an inclusion-wise maximal face, one has $F = \{x \in P \mid a^{\mathsf{T}}x = \beta\}$.

Let F be of the form $F = \{x \in P \mid a^{\mathsf{T}}x = \beta\}$ for an inequality $a^{\mathsf{T}}x \leq \beta$ in $A^{\leq}x \leq b^{\leq}$. Clearly $F \neq \emptyset$, since the system $Ax \leq b$ is irredundant. If F is not a facet, then $F \subseteq F' = \{x \in P \mid a'^{\mathsf{T}}x = \beta'\}$ with another inequality $a'^{\mathsf{T}}x \leq \beta'$ of $A^{\leq}x \leq b^{\leq}$. Let $x^* \in \mathbb{R}^n$ be a point with $a^{\mathsf{T}}x^* > \beta$, which satisfies all other inequalities in $Ax \leq b$. Such an x^* exists, since $Ax \leq b$ is irredundant. Let $\tilde{x} \in P$ with $A^{\leq}\tilde{x} < b^{\leq}$. There exists a point $\overline{x}$ on the line-segment $\overline{\tilde{x}x^*}$ with $a^{\mathsf{T}}\overline{x} = \beta$. This point is then also in F' and thus $a'^{\mathsf{T}}\overline{x} = \beta'$ follows. This shows that $a'^{\mathsf{T}}x^* > \beta'$ and thus $a^{\mathsf{T}}x \leq \beta$ can be removed from the system. This is a contradiction to $Ax \leq b$ being irredundant. ☐

Lemma 8. *A face F of $P(A,b)$ is inclusion-wise minimal if and only if it is of the form $F = \{x \in \mathbb{R}^n \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.*

Proof. Let F be a minimal face of P and let $A'x \leq b'$ be the subsystem of inequalities of $Ax \leq b$ with $F = \{x \in P \mid A'x = b'\}$. Suppose that $F \subset \{x \in \mathbb{R}^n \mid A'x = b'\}$ and let $x_1 \in \mathbb{R}^n \setminus P$ satisfy $A'x_1 = b'$ and $x_2 \in F$. There exists "a first" inequality $a^T x \leq \beta$ of $Ax \leq b$ which is "hit" by the line-segment $\overline{x_2 x_1}$. Let $x^* = \overline{x_2 x_1} \cap (a^T x = \beta)$. Then $x^* \in F$ and thus $F \cap (a^T x = \beta) \neq \emptyset$. But $F \supset F \cap (a^T x = \beta)$ since $a^T x \leq \beta$ is not an inequality of $A'x \leq b'$. This is a contradiction to the minimality of F.

Suppose that F is a face with $F = \{x \in \mathbb{R}^n \mid A'x = b'\} = \{x \in P \mid A'x = b'\}$ for a subsystem $A'x \leq b'$ of $Ax \leq b$. Suppose that there exists a face $\widetilde{F}$ of P with $\emptyset \subset \widetilde{F} \subset F$. By Lemma 6 $\widetilde{F} = \{x \in P \mid A'x = b', A^*x = b^*\}$, where $A^*x \leq b^*$ is a sub-system of $Ax \leq b$ which contains an inequality $a^T x \leq \beta$ such that there exists an $x_1, x_2 \in F$ with $a^T x_1 < \beta$ and $a^T x_2 \leq \beta$. The line $\ell(x_1, x_2) = \{x_1 + \lambda(x_2 - x_1) \mid \lambda \in \mathbb{R}\}$ is contained in F but is not contained in $a^T x \leq \beta$. This shows that F is not contained in P which is a contradiction. □

We say that a polyhedron contains a line $\ell(x_1, x_2)$ with $x_1 \neq x_2 \in P$ if $\ell(x_1, x_2) = \{x_1 + \lambda(x_2 - x_1) \mid \lambda \in \mathbb{R}\} \subseteq P$. A *vertex* of P is a 0-dimensional face of P. An *edge* of P is a 1-dimensional face of P.

Exercise 3.

i) Show that the dimension of each minimal face of a polyhedron P is equal to $n - \text{rank}(A)$.

ii) Show that a polyhedron has a vertex if and only if the polyhedron does not contain a line.

The simplex method walks from vertex to vertex along edges of a polyhedron with vertices.

Integral polyhedra

A rational polyhedron P is called *integral* if each minimal face of P contains an integer point.

Theorem 9. *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a rational nonemty polyhedron with vertices. P is integral if and only if for all integral vectors $c \in \mathbb{Z}^n$ with* $\max\{c^T x \mid x \in P\} < \infty$ *one has* $\max\{c^T x \mid x \in P\} \in \mathbb{Z}$.

Proof. Let P be integral and $c \in \mathbb{Z}^n$ with $\max\{c^T x \mid x \in P\} = \delta < \infty$. Since the face $F = \{x \in P \mid c^T x = \delta\}$ contains an integer point it follows that $\delta \in \mathbb{Z}$.

On the other hand let x^* be a vertex of P and assume that $x^*(i) \notin \mathbb{Z}$. There exists a subsystem $A'x \leq b'$ of $Ax \leq b$ with $A' \in \mathbb{R}^{n \times n}$, A nonsingular and $A'x^* = b'$. Let $a_1, \ldots, a_n$ be the columns of A'. Since A' is invertible, there exists an integer vector $c \in \text{cone}(a_1, \ldots, a_n) \cap \mathbb{Z}^n$ such that $c \pm e_i \in \text{cone}(a_1, \ldots, a_n)$. The point x^* maximizes both $c^T x$ and $(c + e_i)^T x$. Clearly not both numbers $c^T x^*$ and $(c + e_i)^T x^*$ can be integral, which is a contradiction. □

Lemma 10. *Let $A \in \mathbb{Z}^{n\times n}$ be an integral and invertible matrix. One has $A^{-1}b \in \mathbb{Z}^n$ for each $b \in \mathbb{Z}^n$ if and only if* $\det(A) = \pm 1$.

Proof. Recall Cramers rule which says $A^{-1} = 1/\det(A)\widetilde{A}$, where $\widetilde{A}$ is the adjoint matrix of A. Clearly $\widetilde{A}$ is integral. If $\det(A) = \pm 1$, then A^{-1} is an integer matrix.

If $A^{-1}b$ is integral for each $b \in \mathbb{Z}^n$, then A^{-1} is an integer matrix. We have $1 = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1})$. Since A and A^{-1} are integral it follows that $\det(A)$ and $\det(A^{-1})$ are integers. The only divisors of one in the integers are ± 1. □

A matrix $A \in \mathbb{Z}^{m\times n}$ with $m \leq n$ is called *unimodular* if each $n \times n$ sub-matrix has determinant $0, \pm 1$.

Theorem 11. *Let $A \in \mathbb{Z}^{m\times n}$ be an integral matrix of full row-rank. The polyhedron defined by $Ax = b, x \geq 0$ is integral for each $b \in \mathbb{Z}^m$ if and only if A is unimodular.*

Proof. Suppose that A is unimodular and b is integral. The polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$ does not contain a line and thus has vertices. A vertex x^* is of the form $x^*_B = A_B^{-1}b$ and $x^*_{\overline{B}} = 0$, where $B \subseteq \{1, \dots, n\}$ is a basis. Since A_B is unimodular one has $x^* \in \mathbb{Z}^n$.

If A is not unimodular, then there exists a basis B with $\det(A_B) \neq \pm 1$. By Lemma 10 there exists an integral $b \in \mathbb{Z}^n$ with $(A_B)^{-1}b \notin \mathbb{Z}^m$. Let λ be the maximal absolute value of a component of $A_B^{-1}b$. Then $b' = \lceil \lambda \rceil A_B \mathbf{1} + b$ is an integral vector with $A_B^{-1}b' = \lceil \lambda \rceil \mathbf{1} + A_B^{-1}b \geq 0$ and $A_B^{-1}b' \notin \mathbb{Z}^m$. The polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b',\ x \geq 0\}$ has thus a fractional (non-integer) vertex. □

An integral matrix $A \in \{0, \pm 1\}^{m\times n}$ is called *totally unimodular* if each of its square sub-matrices has determinant $0, \pm 1$.

Theorem 12 (Hoffman-Kruskal Theorem)**.** *Let $A \in \mathbb{Z}^{m\times n}$ be an integral matrix. The polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\ x \geq 0\}$ is integral for each integral $b \in \mathbb{Z}^m$ if and only if A is totally unimodular.*

Proof. The polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\ x \geq 0\}$ is integral if and only if the polyhedron $Q = \{z \in \mathbb{R}^{n+m} \mid (A|I)z = b,\ z \geq 0\}$ is integral. The assertion thus follows from Theorem 11. □

Exercise 4. In this exercise you can assume that a linear program $\max\{c^{\mathrm{T}}x \mid Ax \leq b\}$ can be solved in polynomial time. Suppose that $P(A, b)$ has vertices and that the linear program is bounded. Show how to compute an optimal *vertex* solution of the linear program in polynomial time.

If an integral polyhedron has vertices, then an optimal vertex solution of a linear program over this polyhedron is integral.

Bipartite matching

An undirected graph $G = (V, E)$ is a tuple, where V is a finite set and E is a set of unordered pairs of V. The set V is called *nodes* and the set E are the *edges* of G. We write uv in short for the edge $\{u, v\} \subseteq V$. The graph is *bipartite*, if V has a partition into sets A and B such that each edge uv satisfies $u \in A$ and $v \in B$.

A *matching* in G is a subset $M \subseteq E$ such that $e_1 \cap e_2 = \emptyset$ holds for each $e_1 \neq e_2 \in M$. Let $c : E \to \mathbb{R}$ be a weight function. The weight of a matching is defined as $c(M) = \sum_{e \in M} c(e)$. The *weighted matching problem* is defined as follows: Given a graph $G = (V, E)$ and edge-weights $c : E \to \mathbb{R}$, compute a matching M of G with $c(M)$ maximal.

We introduce decision variables $x(e)$ for each edge $e \in E$. We want to model the characteristic vectors $\chi^M \in \{0, 1\}^E$ of matchings, where

$$\chi^M(e) = \begin{cases} 1 & \text{if } e \in M, \\ 0 & \text{if } e \notin M. \end{cases}$$

This is achieved with the following set of constraints.

$$\begin{aligned} \sum_{e \in \delta(v)} x(e) &\leq 1 && \text{for each } v \in V, \\ x(e) &\geq 0 && \text{for each } e \in E. \end{aligned} \tag{7}$$

Clearly, the set of vectors $x \in \mathbb{Z}^E$ which satisfy the system (7) are exactly the characteristic vectors of matchings of G. The matrix $A \in \{0, 1\}^{V \times E}$ which is defined as

$$A(v, e) = \begin{cases} 1 & \text{if } v \in e, \\ 0 & \text{if } v \notin e \end{cases}$$

is called *node-edge incidence matrix* of G.

Lemma 13. *If G is bipartite, the node-edge incidence matrix of G is totally unimodular.*

Lemma 13 implies that each vertex of the polytope P defined by the inequalities (7) is integral. Thus an optimal vertex of the linear program $\max\{c^T x \mid x \in P\}$ corresponds to a maximum weight matching.

Proof of Lemma 13. Let $G = (V, E)$ be a bipartite graph with bi-partition $V = V_1 \cup V_2$.

Let A' be a $k \times k$ sub-matrix of A. We are interested in the determinant of A. Clearly, we can assume that A does not contain a column which contains only one 1, since we simply consider the sub-matrix A'' of A', which emerges from developing the determinant of A' along this column. The determinant of A' would be $\pm 1 \cdot \det(A'')$.

Thus we can assume that each column contains exactly two ones. Now we can order the rows of A' such that the first rows correspond to vertices of V_1 and then follow the rows corresponding to vertices in V_2. This re-ordering only affects the sign of the determinant. By summing up the rows of A' in V_1 we obtain exactly the same row-vector as we get by summing up the rows of A' corresponding to V_2. This shows that $\det(A') = 0$. $\square$

Network flows

Let $G = (V, A)$ be a directed graph. Recall the definition of a flow vector $f \in \mathbb{R}^A$. The *node-edge incidence matrix of a directed graph* is a matrix $A \in \{0, \pm 1\}^{V \times E}$ with

$$A(v,a) = \begin{cases} 1 & \text{if } v \text{ is the starting-node of } a, \\ -1 & \text{if } v \text{ is the end-node of } a, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

A feasible flow f in G with capacities u and in-out-flow b is then a solution $f \in \mathbb{R}^A$ to the system

$$Af = b, \qquad 0 \leq f \leq u.$$

Lemma 14. *The node-edge incidence matrix A of a directed graph is totally unimodular.*

Proof. Let A' be a $k \times k$ submatrix of A. Again, we can assume that in each column we have exactly one 1 and one -1. Otherwise, we expand the determinant along a column which does not have this property. But then, the A' is singular, since adding up all rows of A' yields the 0-vector. $\square$

A consequence is that, if the vector b and the capacities u are integral and an optimal flow exists, then there exists an integral optimal flow. We have seen that this follows from the cycle-cancelling algorithm, but total unimodularity gives another simple and elegant proof of this fact.

Further applications of polyhedra theory

Doubly stochastic matrices

A matrix $A \in \mathbb{R}^{n \times n}$ is *doubly stochastic* if it satisfies the following linear constraints

$$\begin{aligned} \sum_{i=1}^{n} A(i,j) &= 1 \qquad \text{for } j = 1, \ldots, n, \\ \sum_{j=1}^{n} A(i,j) &= 1 \qquad \text{for } i = 1, \ldots, n, \\ A(i,j) &\geq 0 \qquad \text{for } i,j = 1, \ldots, n. \end{aligned} \tag{9}$$

A *permutation matrix* is a matrix which contains exactly one 1 per row and column, where the other entries are all 0.

Theorem 15. *A matrix $A \in \mathbb{R}^{n \times n}$ is doubly stochastic if and only if A is a convex combination of permutation matrices.*

Proof. Since a permutation matrix satisfies the constraints (9), then so does a convex combination of these constraints.

For the converse, it is enough to show that each vertex of the polytope defined by the system (9) is integral and thus a permutation matrix. However, the matrix defining the system (9) is the node-edge incidence matrix of the complete bipartite graph having $2n$ vertices. Since such a matrix is totally unimodular, the theorem follows. □

Bibliography

[1] A. Schrijver. *Theory of Linear and Integer Programming*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley, Chichester, 1986. A Wiley-Interscience publication.