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# Combinatorial Optimization

Fall 2013

Assignment Sheet 3

**Exercise 1**

It is easy to find graphs where, at each contraction of nodes, we have roughly $1/2$ probability of destroying a minimum cut. Consider for instance the graph made of two complete graphs with $n/2$ nodes (say $V_1$ and $V_2$), plus exactly one edge connecting them. If we choose two nodes at random and contract them, then we save the unique minimum cut iff those nodes are chosen from the same set $V_i$ Hence, at the end of the algorithm, we saved the minimum cut with probability roughly $2^{-n}$, which is way less that the $n^{-2}$ guaranteed by Karger's algorithm. To obtain a constant probability of success with this new algorithm, we would need to repeat it an exponential number of times.

**Exercise 2 and 3**

See Sections 10.2.1 and 10.2.2 in *Randomized Algorithms*, R. Motwani and P. Raghavan, Cambridge University Press.

**Exercise 4**

The family $\mathscr{C}$ of cuts induced by removing each edge of a Gomory-Hu tree has this property. Hence, $n-1$ cuts suffices, and they can be found in polynomial time (since a Gomory-Hu tree can be found in polynomial time). To show that it is tight for each $n$, take the complete graph.

**Exercise 5**

Node that the contraction of $s$ and $t$ has no effect on the capacity of a cut that contains both (or none) of $s$ and $t$. Now pick a cut $U$ that divides $s$ from $t$. The capacity of $U$ after the contraction is equal to the capacity of a cut that that does not divide $s$ and $t$. In particular, if the minimum cut does not divide $s$ and $t$, then the contraction preserves its capacity and the fact that it is minimum.

**Exercise 6**

(FYI: The algoritm is due to Nagamochi and Ibaraki. It is remarkable because it is a deterministic algorithm that computes a min-cut without using the relation with max-flow.)

*(a).* We show that in a graph $G$ with a good ordering $v_1, \dots, v_n$, the minimum cut between $v_n$ and $v_{n-1}$ is given by $\{v_n\}$. The correctness of the algorithm will then follow using the

previous exercise. Let $G$ be counterexample to the correctness of the statement above, and suppose that, among all counterexamples, it is one that minimizes $|V| + |E|$. First observe that, in $G$, $v_n$ and $v_{n-1}$ are not adjacent: if yes, remove the edge $(v_{n-1}, v_n)$ as to obtain a new graph $G'$. By minimality of $G$ the statement holds for $G'$, and the ordering is still good, hence the minimum cut between $v_{n-1}$ and $v_n$ in $G'$ is $\{v_n\}$. But then this is a minimum cut between $v_{n-1}$ and $v_n$ (adding the edge $(v_{n-1}, v_n)$ increases the capacity of all cuts separating $v_{n-1}$ and $v_n$ by 1), a contradiction.

Using this fact and again the minimality of $G$, one easily shows that the minimum cut between $v_{n-2}$ and $v_{n-1}$ (resp. $v_n$) in $G$ is $\{v_{n-1}\}$ (resp. $\{v_n\}$): this time, you have to analyze graph $G \setminus \{v_n\}$ (resp. $G \setminus \{v_{n-1}\}$). Then we use the fact that, given any three nodes in a graph, say $u, v, z$, the capacity of the minimum cut that separates $u$ from $v$ is at least the minimum of the capacity of the minimum cuts separating $u$ from $w$ and $w$ from $v$. The thesis follows by taking $u = v_n$, $v = v_{n-1}$ and $w = v_{n-2}$ and observing that $|\delta(\{v_{n-1}\})| \geq |\delta(\{v_n\})|$ by definition of good ordering.

*(b)* From what argued in part (a), we just need to remember the sequence of contractions, and pick the one where the value $x^i$ is minimum.

**Exercise 7**

Consider an ordering $v_1, \ldots, v_n$ of the vertices of a graph. A path $v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_k}$ is called *half-monotone* if there exists $j$ such that $\ell_1 > \ell_2 > \cdots > \ell_j < \ell_{j+1} < \cdots < \ell_k$. Also, we say that the *predecessor* of a node $v_i$ is the node with highest index $j < i$ that is adjacent to $v_i$.

We prove that in a graph $G(V, E)$ with a good ordering $v_1, \ldots, v_n$ such that the degree of $v_n$ is at least $d$, there are $d$ edge-disjoint half-monotone paths between $u = v_n$ and $v = v_{n-1}$. The proof is by double induction on $n$, and then on $d$. Draw the vertices on a line from left to right, following the ordering. Let $v'$ (resp. $u'$) be the leftmost adjacent of $v$ (resp. to $u$). If $v' = u$, then $u$ and $v$ are adjacent: the edge connecting them gives one path (that is clearly half-monotone), and either $d = 1$ and we are done, or we can apply induction on the graph obtained by removing $uv$ ($d$ has decreased by 1 and the good ordering stays the same). If $u' = v'$, then we can remove the edges $uu', u'v$ from $G$, and argue as above. So suppose $u' = v_j < v'$ wrt the good ordering (the case $v' < u'$ is analogous). Consider the sequence of nodes $v''$, $v'''$, etc. obtained by taking the predecessor of $v'$, $v''$, etc. while the predecessor lies strictly after $u'$. Call the last such node $\bar{v}'$. Note that $\bar{v}'$ has at least 1 edge pointing to vertices $v_1, \ldots, v_j$. In fact, $d^j(v') \geq d^j(v_n) = 1$, where $d^i(z)$ is the number of nodes adjacent to $z$ with indices in $1, \ldots, i$. Hence, we can apply induction and conclude that, in the subgraph of $G$ induced by $v_1, \ldots, v_j, \bar{v}'$ there is one half-monotone path between $u'$ and $\bar{v}'$ (again, the ordering induced is a good order). We can then immediately extend this path to a half-monotone path between $u$ and $v$, and apply induction on $d$ (one easily checks that there exists a good ordering of the new graph where $v_{n-1}$ and $v_n$ are still last).