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Combinatorial Optimization

Fall 2013

Assignment Sheet 3

Exercise 1
Calculations are left to the reader.

Exercise 2
It is easy to check that the problem are equivalent if costs are nonnegative.

Exercise 3

(a). The stronger statement is true. In order to prove it, take any spanning tree with at least one edge uv of length $2d$ that does not belong to the Delaunay triangulation. Note that there exists no point $z \in \mathbb{R}^2$ such that $||u-z||_2 = ||v-z||_2 < ||w-z||_2$ for each $w \in V \setminus \{u, v\}$. This is clear if $P_v \cap P_u = \emptyset$. If $|P_v \cap P_u| = 1$, it follows by continuity. Now let z be the middle point of u and v, and let $w \in V \setminus \{u, v\}$ be a vertex of minimum distance from z. From what argued above, $||w-z||_2 \leq ||u-z||_2 = ||v-z||_2 = d$. That is, z is the center of a circle of diameter $2d$ that contains w, u, and v, with the latter two being opposite points. This implies that $||w-u||_2, ||w-v||_2 < 2d$. Now assume wlog that w is in the same connected component of u in the graph $T' := T \setminus \{uv\}$. It is easy to check that $T' \cup \{wv\}$ is a tree, and from what argued above it has a smaller cost than T.

(b). From (a), any minimum spanning tree of G is a minimum spanning tree of G', hence we can focus on finding one of the latter. Observe that G' is a planar graph. In order to show this, first observe that P_u and P_v share a segment disjoint from all other P_w and that both P_u and P_v are connected (both those statements are easy to check, but you may want to explicitly prove them). Hence, we can draw the edge between u and v by going from u to the relative interior of the shared segment, and from there to v. By construction, no two of those edges intersect, hence G' is planar. It is well-known that a planar graph has $O(|V|)$ edges (this can be easily derived e.g. using Euler's formula). Using part (d) from Exercise 8 of the first sheet, we deduce that we can compute a minimum spanning tree in G in time $O(|V|\log|V|)$. (FYI, it can be shown that the same time bound in fact suffices to compute the graph G', hence we can find a minimum spanning tree in the plane in time $O(|V|\log|V|)$).

(c). This question could be interpreted in two ways. I accepted (and discuss below) both.

First interpretation: Both c and the Voronoi Diagram are now wrt to a generic norm. The stronger version of the statement is false. As a counterexample, take the four points $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$, and the ℓ_1 norm (i.e. $||(x_1, y_1)||_1 = |x_1| + |y_1|$). The distance between any two of those points is 2, hence any spanning tree is optimum, but one easily checks that in the Voronoi Diagram, $|P_{(0,0)} \cap P_{(2,0)}| = 1$.

For the weaker statement, the argument of (a) can be repeated for generic norms up to $||w - z|| \leq ||u - z|| = ||v - z|| = d$. The triangular inequality implies that $||w - v|| \leq 2d$. Let T' be obtained as in (a): we already know that the cost of $T' \cup \{vw\}$ is less or equal the cost of T, and it is strictly less if and only if $||w - v|| < 2d$. If $||w - v|| < 2d$ we are done, so suppose $||w - v|| = 2d$, hence $T' \cup \{vw\}$ has the same cost of T. If $wv \in E(G')$ we are done, else we repeat the argument, replacing the edge wv with a new edge, until we find an edge of G'. Note that an edge, once discarded, will not be examined again. In fact, suppose e.g. that z' is the middle point of wv. Again, we take $w' \in V \setminus \{w, v\}$ such that $||w' - z'||$ is minimized, and we know that $||w' - z'|| \leq ||w - z'||, ||v - z'||$. Note that $w' \neq u$, else P_u and P_v share two points, a contradiction. The statement then follows from the finiteness of V.

Second Interpretation: The Voronoi Diagram stays the same, but c changes. Both statements are false. Consider the point $(0,0)$,$(1,0)$,$(\sqrt{2}/2, \sqrt{2}/2)$, and $(\sqrt{2}/2, -\sqrt{2}/2)$. In the Voronoi Diagram (wrt the Euclidean norm) we have that $|P_{(0,0)} \cap P_{(1,0)}| = 1$, but you can easily verify that all minimum spanning trees wrt to the ℓ_1 norm take the edge between $(0,0)$ and $(1,0)$.

Exercise 4

(a) [2.] ⇒ [1.]: It can be easily shown by induction on $|S \triangle T|$.
[1.] ⇒ [3.]: Immediate.
[3.] ⇒ [2.]: It can be easily shown by induction on $|T \setminus S|$.

(b) Add to [1.] the condition that f is non-decreasing, i.e. $f(T) \geq f(S)$ for all $S \subseteq T \subseteq I$.

We show [1′.] ⇒ [2′.]. Pick $S \subseteq T \subseteq I$. We know [2′.] holds true for $x \notin T$ (it is equivalent to [2.], which we showed follows from [1.]). If $x \in S$ then [2′.] trivially holds. So let $x \in T \setminus S$. We have

$$f(T \setminus x) - f(T) = 0 \leq f(S \cup x) - f(S)$$

as required, where the last inequalities holds because the function is non-decreasing. The reverse argument shows that [2′.] ⇒ [1′.].

Exercise 5

We answer the question for cases f of g, which are the most interesting and general.

The rank function of a matroid is submodular. Let $(M, \mathscr{I})$ be a matroid, $X, Y \subseteq M$. We show that $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$. Let I be a maximum independent set of $X \cap Y$, and let I' its extension to a maximum independent set of X, and I'' the extension of the latter to a maximum independent set of $X \cup Y$. We need to show that $r(Y) \geq |I| + |I''| - |I'|$. Note that $|I'' \cap Y| = |I'' \setminus (I' \setminus Y)| = |I'' \setminus (I' \setminus I)|$, hence we deduce

$$r(Y) \geq |I'' \cap Y| = |I'' \setminus (I' \setminus I)| = |I''| - |I'| + |I|$$

where the first inequality followed from the fact that $I'' \cap Y$ is contained in Y, and it is a subset of an independent set, hence independent; the second from what argued above; the last from the fact that $I \subseteq I' \subseteq I''$.

The Entropy of a set of random variables is submodular. The *conditional entropy* of two random variables X and Y with possible values respectively $x_1, \dots, x_k$ and $y_1, \dots, y_{k'}$ is $\sum_{i,i'} P(X = x_i, Y = y_{i'}) \log \frac{p(Y=y_{i'})}{p(X=x_i, Y=y_{i'})}$. This generalizes in the obvious way when X and Y are sets of random variables. One readily verifies that, for $A \subseteq B$ and $i \notin B$, one has

$$H(\{i\}|B) - H(B \cup \{i\}) = -H(B) \quad \text{and similarly} \quad H(\{i\}|A) - H(A \cup \{i\}) = -H(A),$$

and that moreover $H(\{i\}|B) \geq H(\{i\}|A)$. We deduce

$$H(A \cup \{i\}) - H(A) = H(\{i\}|A) \leq H(\{i\}|B) = H(B \cup \{i\}) - H(B),$$

as required.

Exercise 6

In class, we deduced from Karger's algorithm that $\binom{n}{2} = n(n-1)/2$ is an upper bound to the number of minimum cuts in a graph. To see it is tight, take the circuit with all edge weights one. Call its nodes $v_1, \dots, v_n$. A cut here is minimum if and only if it takes consecutive nodes, so we need to count those. We associate each of them with a distinct pair of nodes (i, j), with $i < j$ as follows: for $i > 1$, set $S = \{v_i, v_{i+1}, \dots, v_j\}$. For $i = 1$, take $S = \{j\}$. One easily checks that the cuts are different, and minimum. Since the possible choices of pairs of nodes as above are $\binom{n}{2}$, the thesis follows.