Prof. Friedrich Eisenbrand
Dr. Yuri Faenza

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Combinatorial Optimization

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Assignment Sheet 1

For an independence system $(S, \mathscr{I})$, define

0. if $I, J \in \mathscr{S}$ with $|J| > |I|$, then there exists $x \in J \setminus I$ such that $I \cup \{x\} \in \mathscr{S}$.

Exercise 1

For the first one, we set $S = E(G)$ and $\mathscr{I} = \{I \subseteq S : I \text{ is a a subset of an } s-t \text{ path}\}$. For the other two we proceed similarly. It is easy to provide examples where some of the equivalent matroid conditions 0-3 are not satisfied, hence none of them is a matroid.

Exercise 2

Assuming $(S, \mathscr{I})$ is an independence system, it is enough to show that 0 holds true if and only if any of $1-3$ holds true.

$0 \Longleftrightarrow 1$ Clear.

$0 \Longleftrightarrow 2$ One direction is clear. For the other, assume 2 holds true. We prove the statement by induction on $|I \setminus J|$. When $|I \setminus J| = 0$, then $I \subseteq J$ and the statement holds by definition of independence system. So suppose $|I \setminus J| \geq 1$. Pick $i \in I \setminus J$. We apply induction on $I \setminus \{i\}$ and J, as to obtain $j \in J \setminus I$ such that $I \cup \{j\} \in \mathscr{I}$. We apply induction again on $I \setminus \{i\} \cup \{j\}$ and J, as to obtain an element $j' \in J \setminus \{I, j\}$ with $I \setminus \{i\} \cup \{j, j'\} \in \mathscr{I}$. We now apply 2 to I and $I \setminus \{i\} \cup \{j, j'\}$, and deduce that either $I \cup \{j\}$ or $I \cup \{j'\} \in \mathscr{I}$. Since $j, j' \in J \setminus I$, we deduce the thesis.

$0 \Longleftrightarrow 3$ One direction is clear. For the other, assume 3 holds true. Then if $I, J \in \mathscr{I}$ and $|I| < |J|$, I cannot be maximal. Suppose by contradiction that $I' = I \cup \{x\} \in \mathscr{I}$ for some $x \setminus (J \cup I)$, but for no $y \in J \setminus I$. Then condition 3 is violated when we pick $A = J \cup \{x\}$, since both J and I are maximum independent sets within A, but they are of different cardinality by assumption.

Property 3 is often very useful when we want to show that an independence system is a matroid. We will use it frequently in the next exercises.

Exercise 3

Clearly $(S, \mathscr{I})$ is an independence system. It is then enough to verify condition 3 from Exercise 2 holds true. Suppose not, i.e. there exist some set A with two bases $I, J \subseteq A$ of different cardinality, say $|I| < |J|$. Clearly $|I \setminus J|, |J \setminus I| \neq \emptyset$, else the statement is trivial. Let M_1 and M_2 be minimal matchings that cover vertices of I and J respectively. Consider the graph $G' = (V, M_1 \triangle M_2)$. Nodes with degree one are those covered by exactly one of M_1 and M_2, while nodes with degree 2 (resp. 0) are those covered by both matchings with different edges (resp. the same edge). Hence G' is the disjoint union of even cycles, paths where edges alternate between M_1 and M_2, and isolated vertices. Since $|J \setminus I| > |I \setminus J|$, there exist some node of J, say v, that is not matched to any node of I, and that is the extreme point of a path of G' that ends in a vertex $u \notin I$. This path P has one edge from M_2 more than it has edges from M_1. Then the matching $M_1' = M_1 \triangle M_2$ covers nodes $I \cup \{v\} \subset A$, contradicting the fact that I is a basis.

Exercise 4
Clearly M is an independence system. We show that condition 3 from Exercise 2 holds. Take $A \subseteq E$, and any basis I of A (wrt M). Then $I \cap S_1 \in \mathscr{I}_1$. It is in fact a basis of A (wrt M_1), otherwise there exists $x \in (S \cap A) \setminus I_1$ such that $I \cup \{x\} \subseteq S$ is independent in M, a contradiction. So $r_1(A \cap S_1) = |I \cap S_1|$ and similarly $r_2(A \cap S_2) = |I \cap S_2|$. Hence $|I| = r_1(A \cap S_1) + r_2(A \cap S_2)$, hence it does not depend on J. Hence $r(A) = r_1(A \cap S_1) + r_2(A \cap S_2)$ is independent of the basis of A we chose, and we are done. Note that this proof technique lets us, at the same time, prove that an independence system is a matroid *and* describe its rank function. This will also be useful for next exercises.

Exercise 5
The *uniform matroid of rank t over n elements* $M = (E, \mathscr{I})$ is such that $E = \{1, \ldots, n\}$ and $\mathscr{I} = I \subseteq E$ such that $|I| \leq t$. It is easy to see that it is a matroid. The partition matroid can be seen as the union of the rank 1 uniform matroids over E_i. Using the result from the previous exercise, we conclude it is a matroid.

Exercise 6
If an independence system is not a matroid, there exists $A \subseteq E$ that violates property 3 from Exercise 2. In particular, let I be a maximal independent set contained in A not of maximum cardinality, while let J be a maximal independent set of A of maximum cardinality. Let $c_e = 1 + \varepsilon$ for $e \in I$, $c_e = 1$ for $e \in A \setminus I$, and $c_e = 0$ otherwise, where $\varepsilon > 0$ to be fixed. Then the greedy algorithm will output I. For ε small enough (e.g., $\varepsilon < 1/|I|$), we have

$$c(J) = |J| \geq |I| + 1 > |I|(1 + \varepsilon) = c(I),$$

hence I is not an independence system of maximum weight.

Exercise 7
Take a smallest independent set J such that $J \cup \{x\}$ contains two circuits C_1 and C_2 for some $x \notin J$. By the minimality of J we deduce that $J \cup \{x\} = C_1 \cup C_2$. Moreover, there exist a, b with $a \in C_1 \setminus C_2$ and $b \in C_2 \setminus C_1$ (else one is contained in the other, contradicting the fact that they are circuit). Let $J' = C_1 \cup C_2$ and note that $J' \setminus \{a, b\}$ is an independence set, since otherwise it would contain a circuit C, and $J \setminus \{a\} \cup \{x\}$ would contain the circuits C_2 and C, contradicting the minimality of J. Hence $J' \setminus \{a, b\}$ and J are maximal independent sets contained in $J \cup \{x\}$, but they have different cardinalities, contradicting property 3 from Exercise 2.

Exercise 8
M^* is usually called *dual matroid*. It is easy to check that it is indeed an independence

system. We now show it satisfies property 3 from Exercise 2, with an approach similar to the one used in Exercise 4, i.e. picking a maximal independent set J^* (wrt M^*) of some $A \subseteq S$ and showing that $|J^*|$ is independent of the specific J^* we picked. This will also characterize the rank function of M^*. Hence, let A and J^* be as above. This means that $r(S \setminus J^*) = r(S)$. Consider a basis B (wrt M) of $S \setminus A$ and extend it to a basis B' (again wrt M) of $S \setminus J^*$ (this can be done since M is a matroid, hence satisfies property 0). We claim that $A \setminus J^* \subseteq B'$. Suppose not, then $x \notin B'$ for some $x \in A \setminus J^*$. Then

$$r(S) = r(S \setminus J^*) = r(B') = r(S \setminus (J^* \cup \{x\})),$$

contradicting the fact that J^* is an independent set of maximum size (wrt M^*) contained in A. Hence,

$$r(S) = r(S \setminus J^*) = |B'| = |B| + |A \setminus J^*| = r(S \setminus A) + |A| - |J^*|,$$

from which we deduce that $|J^*| = r(S \setminus A) - r(S) + |A|$, hence does not depend on J^*, as required.