**Fall 2014 Randomized Algorithms** Oct 8, 2014

# Lecture 3

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In this lecture we will be concerned with linear programming, in particular Clarkson's Las Vegas algorithm [1]. The main result is an expected polytime reduction from an LP in dimension $d$ with $n$ constraints to $O(n)$ LPs in dimension $d$ with $O(d^2)$ constraints. In a second part, we introduce bounds on the diameter of a polyhedra , a topic which we will view in more detail in the next lecture. Finally, we discuss the smallest enclosing disk problem as an application of Clarkson's algorithm.

## Linear Programming

We consider the traditional formulation of linear programming in $d$ dimensions with $n$ constraints as

$$\begin{aligned} \max_{x \in \mathbb{R}^d} \quad & c^\top x \ , \\ \text{subject to} \quad & Ax \leq b. \end{aligned}$$

Throughout this lecture, we will make the following assumptions:

- $A \in \mathbb{R}^{n \times d}$ has full rank.
- There exists a basis $B \subseteq \{1, \ldots, n\}$, $|B| = d$ such that $A_B$ is non singular and $x^* = A_B^{-1} \cdot b_B$ is optimal ($x^*$ has $d$ tight constraints).
- The optimal solution $x^*$ is unique (note that this can always be achieved by slightly perturbing the objective function).

## Clarkson's Algorithm

We begin with a very useful lemma. We remind that a *multiset* is a set in which members may appear more than once. We denote the cardinality of a multiset $H$ by $\mu(H)$. By $\binom{H}{r}$, we denote the collection of all 'sub-multisets' of $H$ of cardinality $r$. For a set (or multiset) $U$ of constraints, let $x^*_U$ be the optimal solution (if it exists) of the LP restricted to the constraints in $U$.

**Lemma 1 (Sampling Lemma)** *Let $G$ be a set of constraints and $H$ be a multiset of constraints of dimension $d$. Let $R \in \binom{H}{r}$ be picked uniformly at random. Let $V_R = \{h \in H \ : \ x^*_{G \cup R}$ violates $h\}$. Then, $E\left[|V_R|\right] \leq d \cdot \frac{\mu(H)-r}{r+1}$.*

**Proof** We have $E[|V_R|] = \sum_{R \in \binom{H}{r}} \frac{1}{\binom{\mu(H)}{r}} |V_R|$. Define the characteristic function

$$\chi(h, R) = \begin{cases} 1 & \text{if } x^*_{G \cup R} \text{ violates } h \\ 0 & \text{otherwise.} \end{cases}$$

We get

$$\begin{aligned} E[|V_R|] &= \sum_{R \in \binom{H}{r}} \frac{1}{\binom{\mu(H)}{r}} \sum_{h \in H \setminus R} \chi(h, R) \\ &= \frac{1}{\binom{\mu(H)}{r}} \sum_{Q \in \binom{H}{r+1}} \sum_{h \in Q} \chi(h, Q - h), \end{aligned}$$

where the last equality holds since picking $r$ constraints and then an additional constraint from $H$ is equivalent to picking $r + 1$ constraints from $H$ and then one constraint out of these.

Fix a basis $B$ of $G \cup Q$. Suppose that the chosen constraint $h$ is violated by $x^*_{G \cup (Q-h)}$ (and thus $\chi = 1$). Then, it follows that $h$ has to be part of the basis $B$. Since $|B| = d$, we have $\chi = 1$ for at most $d$ constraints. Thus,

$$E[|V_R|] = \frac{1}{\binom{\mu(H)}{r}} \sum_{Q \in \binom{H}{r+1}} d = d \cdot \frac{\binom{\mu(H)}{r+1}}{\binom{\mu(H)}{r}} = d \cdot \frac{\mu(H) - r}{r + 1}.$$

■

We can now present the **Clarkson(2)** algorithm. The **Clarkson(1)** algorithm will be discussed in the next lecture.

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**Algorithm 1** Clarkson(2)

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**Input:** A multiset $H$ containing the $n$ constraints of the LP
**Output:** The solution $x^*$ to the LP

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Set r = 6 · d^2
repeat
    Pick R ∈ (H choose r) uniformly at random
    Compute x*_R and V_R = {h ∈ H : x*_R violates h}
    if |V_R| ≤ (1/(3d)) μ(H) then
        H = H + V_R (double the occurrence of each h ∈ V_R in H.
    end if
until V_R = ∅
return x*_R
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**Analysis of Clarkson(2)** First of all, note that we reduce the problem of solving our LP with $n$ constraints to the problem of solving LPs with $r = 6 \cdot d^2 = O(d^2)$ constraints (line 4). We thus still need to show that the whole procedure is repeated $O(n)$ times.

We analyse the probability that the *if* statement (line 5) is satisfied. By the Sampling Lemma, we have

$$E[|V_R|] \leq d \cdot \frac{\mu(H) - 6d^2}{6d^2 + 1} \leq \frac{\mu(H)}{6d}.$$

Thus, using Markov's inequality, we get

$$\Pr\left[|V_R| \leq \frac{\mu(H)}{3d}\right] = 1 - \Pr\left[|V_R| > \frac{\mu(H)}{3d}\right] = 1 - \underbrace{\Pr\left[|V_R| > 2 \cdot E[|V_R|]\right]}_{\leq \frac{1}{2}} \geq \frac{1}{2}.$$

We now bound the number of *successful* iterations, where the *if* statement is satisfied (and thus $|V_R| \leq \frac{1}{3d}\mu(H)$). Note that from the previous result, we may then bound the expected number of iterations (and thus the expected number of LPs to solve) by twice the number of successful iterations.

**Lemma 2** *The number of successful iterations is* $O(d \cdot \log(n))$.

**Proof** Let $B \subseteq H$ be the basis for the optimal solution $x^*$ of the LP (note that $B$ is a sub-multiset of $H$ so it may contain a constraint more than once). Let $\mu_i(B)$ denote the cardinality of $B$ after iteration $i$. By definition of the basis, we have $\mu_0(B) = d$.

Note that if $x_R^*$ doesn't violate any constraint of the basis, then it is an optimal solution to the LP. For all but the last iteration, we have $V_R \neq \emptyset$, meaning that $x_R^*$ is not the overall-optimal solution, and thus $V_R$ contains at least one constraint of $B$. Thus, at each iteration, at least one basis constraint has its number of occurrences doubled in $H$. We thus get that $\mu_{i \cdot d}(B) \geq 2^i \cdot d \geq 2^i$.

Now consider how the cardinality of $H$ grows. We have $\mu_0(H) = \mu(H)$. On a successful iteration, we add $|V_R| \leq \frac{1}{3d}\mu(H)$ constraints to $H$. Thus, $\mu_{i+1}(H) \leq \mu_i(H) + \frac{1}{3d}\mu_i(H) = (1 + \frac{1}{3d})\mu_i(H)$. After $i \cdot d$ iterations, we have $\mu_{i \cdot d}(H) \leq (1 + \frac{1}{3d})^{i \cdot d}\mu(H) \leq e^{\frac{i}{3}}\mu(H)$.

Finally, note that the multiset $B$ is always a sub-multiset of $H$. After $d \cdot i$ iterations, we thus must have $2^i \leq n \cdot e^{\frac{i}{3}}$, and therefore $i = O(\log(n))$. We therefore get that the number of successful iterations is $O(d \cdot \log(n))$. ■

Since the probability of an iteration being successful is greater than $\frac{1}{2}$ we expect to have at most $2 \cdot O(d \cdot \log(n)) = O(d \cdot \log(n))$ iterations. Clarkson's algorithm thus reduces an LP in $d$ dimensions with $n$ constraints to an expected $O(d \cdot \log(n))$ LPs with $O(d^2)$ constraints in expected polynomial time.

**A note about unbounded LPs** To deal with LPs that might be unbounded, we can fix a 'box' of $2 \cdot d$ constraints, thus constraining the optimal solutions inside this box. In the analysis, these $2 \cdot d$ constraints are added to the original set $G$ defined in the Sampling Lemma, which we have considered to be empty in the above analysis.

## Open problem: bounding the diameter of a polyhedron

Consider a polyhedron in $d$ dimensions defined by $P = \left\{x \in \mathbb{R}^d \ : \ Ax \leq b\right\}$, where $A \in \mathbb{R}^{n \times d}$. We make the assumption that $P$ is *non-degenerate*, meaning that each vertex of the polyhedron is defined by $d$ tight constraints. For a vertex $x^*$ of $P$, we define the basis $B_{x^*}$ as the set of $d$ tight constraints defining $x^*$. Two vertices $x^*$ and $y^*$ are *neighbours* if $|B_{x^*} \cap B_{y^*}| = d - 1$ (the two vertices only differ in one tight constraint).

**Definition 3** *For a polyhedron $P$, consider the graph $G$ on vertices of $P$ obtained by inserting edges between neighbouring vertices. The* diameter *of $P$ is the largest shortest-path distance (in number of edges) between any two vertices in $G$.*

**Definition 4** *Let $\Delta_{d,n}$ be the largest diameter of a polyhedron $P \subseteq \mathbb{R}^d$ with $n$ constraints.*

**Theorem 5 ([2, 3, 4])**[1] $\Delta_{d,n} \leq n^{\log(d)+1}$.

**Conjecture 6** *There exists a function $f$ such that $\Delta_{d,n} = O(n) \cdot f(d)$ (for a $f$ which does not depend on $n$).*

# 1 Smallest enclosing disk

Consider a set of $n$ points in $\mathbb{R}^2$ and the task of finding the smallest disk enclosing all of these points. First of all, we discuss how we may convince ourselves that a given circle is indeed the smallest enclosing disk.

For a given circle, take all the points appearing on the boundary of the circle. Now, if the centre of the circle is not inside of the *convex hull* of these points, it is easy to see that we may 'shift' the centre of the circle towards the convex hull and shrink it since it now englobes all the points which previously were on its boundary.

One can actually find that a disk is the smallest enclosing disk if it contains all points and one of the following two conditions is satisfied

[1] In [4], Kalai and Kleitman provide an upper bound of $n^{\log(d)+2}$. The tighter bound was shown by Kalai [3] and Eisenbrand et al. [2].

- There exist 3 points on the boundary of the disk such that the centre of the disk lies in the convex hull of the three points.
- There are 2 points on the boundary such that the centre of the disk is their midpoint.

We can verify that the Clarkson(2) algorithm we described can be used to find the smallest enclosing disk in expected $O(n \log n)$ time. Indeed, in each iteration we pick a set $R$ of $6 \cdot d^2 = 24$ points, find a smallest enclosing disk on $R$, check which remaining points fall outside of the disk and double their weight. By a similar analysis as for the LP, we can show that the number of iterations is $O(\log n)$ and that each iteration takes time $O(n)$ (finding the optimal disk on $R$ can be done in constant time since $R$ has constant size; checking which points fall outside the disk can be done in $O(n)$ time through some clever manipulation of copies of points in the multiset).

# References

[1] Kenneth L. Clarkson. Las vegas algorithms for linear and integer programming when the dimension is small. *J. ACM*, 42(2):488–499, March 1995.

[2] Friedrich Eisenbrand, Nicolai Hähnle, and Thomas Rothvoß. Diameter of polyhedra: Limits of abstraction. In *Proceedings of the Twenty-fifth Annual Symposium on Computational Geometry*, SCG '09, pages 386–392, New York, NY, USA, 2009. ACM.

[3] Gil Kalai. Linear programming, the simplex algorithm and simple polytopes. *Math. Programming*, 79:217–233, 1997.

[4] Gil Kalai and Daniel J Kleitman. A quasi-polynomial bound for the diameter of graphs of polyhedra. *Bulletin of the American Mathematical Society*, 26(2):315–316, 1992.