Strong relaxations

for discrete optimization problems

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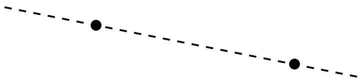
Affine independence and dimension

Let $S \subseteq \mathbb{R}^n$.

Affine hull of S: $\text{aff}(S) = \{x = \sum_{s_i \in S} \alpha_i s_i : \sum_i \alpha_i = 1\}$.



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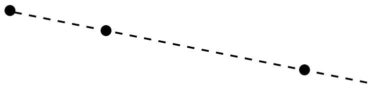

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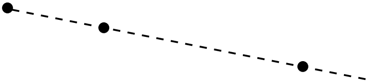
$dim(S) = \max\{|C| : C \subseteq S$ and C is an affinely independent set$\} - 1$.

Obs. $\{v^1, \ldots, v^k\}$ affinely indep $\Leftrightarrow \{v^k - v^1, \ldots, v^2 - v^1\}$ lin indep.

$dim(S) \in \{-1, \ldots, n\}$.



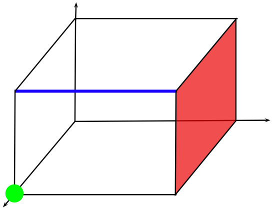
Faces of a polyhedron

Let $cx \leq \delta$ be a valid inequality for a polyhedron $P \subseteq \mathbb{R}^n$.

Then $F = \{x \in P : cx = \delta\}$ if a face of P

and $\{x : cx = \delta\}$ its supporting hyperplane.

P and $\emptyset$ are both (improper) faces.



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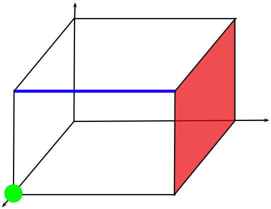

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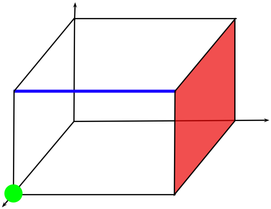

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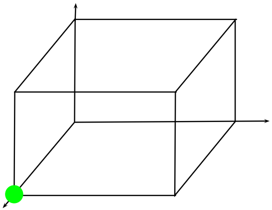
A polyhedron is pointed if it has at least a vertex.

Characterization of vertices

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a pointed polyhedron and $\bar{x} \in P$. TFAE:

- $\bar{x}$ is a vertex of P.
- $\bar{x}$ satisfies at equality n linearly independent inequalities from $Ax \leq b$.
- $\bar{x}$ is not a convex combination of any two other points in P.

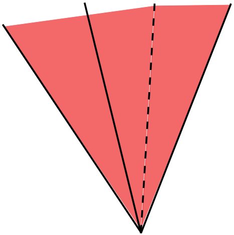


Characterization of extreme rays

A *ray* of a cone C is a set $cone(r) \subseteq C$, for some $r \in \mathbb{R}$.
Edges of a cone are called extreme rays.

Let $C = \{x \in \mathbb{R}^n : Ax \leq 0\}$ be a pointed cone and $\bar{r}$ a ray of C. TFAE:

- $\bar{r}$ is an extreme ray of P.
- $\bar{r}$ satisfies at equality $n-1$ linearly independent inequalities from $Ax \leq 0$.
- $\bar{x}$ is not a conic combination of any two other distinct rays in C.



Characterization of faces

Let $P = \{x \in \mathbb{R}^n : a^i x \leq b_i, i \in [m]\} \neq \emptyset$.

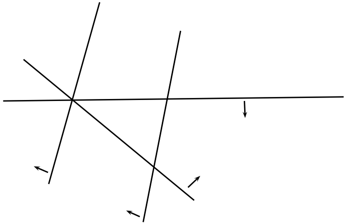
Let $I \subseteq [m]$. The set
$F_I = \{x \in \mathbb{R}^n : a^i x = b_i \text{ for } i \in I, a^i x \leq b_i \text{ for } i \in [m] \setminus I\}$
is a face of P.

Conversely, if F is a non-empty face of P, then $F = F_I$ for some $I \subseteq [m]$.

Implicit equations

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.

An inequality $ax \leq \beta$ from $Ax \leq b$ is an implicit equation if $ax = \beta$ for all $x \in P$.



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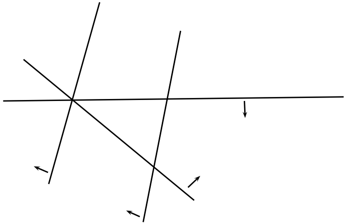

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The system of remaining inequalities is denoted by $A^< x \leq b^<$.

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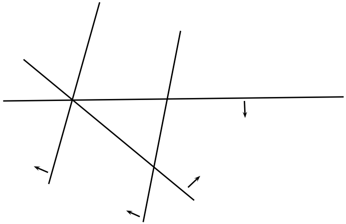

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Obs. There exists $x \in P$ such that $A^{<}x < b^{<}$.

Proof. Take the middle point of points of P, each satisfying strictly one of the inequalities from $A^{<}x \leq b^{<}$.