

Exercises

Optimization in Finance

Fall 2008

Sheet 4

Exercise 4.1

Consider a lock-box problem, where c_{ij} is the cost of assigning region i to a lock-box in region j for $j = 1, \dots, n$. Suppose we wish to open exactly q lock-boxes where $q \in \{1, \dots, n\}$ is a given integer.

- (i) Formulate as an integer program the problem of opening exactly q lock-boxes so as to minimize the total cost of assigning each region to an open lock-box.
- (ii) Formulate in two different ways the constraint that regions cannot send checks to closed lock-boxes.
- (iii) For the following data

$$q = 2 \quad \text{and} \quad (c_{ij})_{1 \leq i, j \leq 5} = \begin{pmatrix} 0 & 4 & 5 & 8 & 2 \\ 4 & 0 & 3 & 4 & 6 \\ 5 & 3 & 0 & 1 & 7 \\ 8 & 4 & 1 & 0 & 4 \\ 2 & 6 & 7 & 4 & 0 \end{pmatrix}$$

compare the linear programming relaxations of your two formulations in question (ii).

Exercise 4.2

Solve the following integer linear program with Branch & Bound.

$$\begin{aligned} & \min(1, -2)x \\ & \begin{pmatrix} -4 & 6 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 9 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \\ & x \in \mathbb{Z}^2 \end{aligned}$$

You can solve the LP subproblems with any computer algebra system.

Exercise 4.3

In this exercise we develop a method, which computes a cutting plane, that cuts off optimal non-integer solutions.

Consider an integer linear program of the form

$$\begin{aligned} \min c^T x & \quad (IP) \\ Ax & = b \\ x & \geq \mathbf{0} \\ x & \in \mathbb{Z}^n \end{aligned}$$

Suppose we use the simplex algorithm to compute a solution x^* to the LP relaxation (that means at least $Ax^* = b$ and $x^* \geq \mathbf{0}$). Let B be the optimal basis (i.e. $A_B x_B^* = b$, $x_{\bar{B}}^* = \mathbf{0}$). Assume there is index $i \in B$ with $x_i^* \notin \mathbb{Z}$. Since $Ax = b$ is a feasible equation for any solution x , also

$$x_B + A_B^{-1} A_{\bar{B}} x_{\bar{B}} = \underbrace{A_B^{-1} A_B}_{=I} x_B + A_B^{-1} A_{\bar{B}} x_{\bar{B}} = A_B^{-1} Ax = A_B^{-1} b \quad (1)$$

holds for any feasible x . Abbreviate β as the i th entry of $A_B^{-1} b$ and let d be the i th row of $A_B^{-1} A_{\bar{B}}$. Then extracting the i th equation from (1) yields that

$$x_i + \sum_{j \in \bar{B}} d_j x_j = \beta$$

holds for any x with $Ax = b$. The *Gomory cut* is now the inequality $x_i + \sum_{j \in \bar{B}} \lfloor d_j \rfloor x_j \leq \lfloor \beta \rfloor$. Prove the following

1. The Gomory cut inequality holds for any solution x to (IP) (that means for any $x \in \mathbb{Z}_+^n$ with $Ax = b$).
2. The cut inequality does not hold for the fractional solution $x^* \notin \mathbb{Z}^n$.
3. An optimum solution to the LP relaxation of

$$\begin{aligned} \min(1, -2)x \\ \begin{pmatrix} -4 & 6 \\ 1 & 1 \end{pmatrix} x & \leq \begin{pmatrix} 9 \\ 4 \end{pmatrix} \\ x & \geq \mathbf{0} \\ x & \in \mathbb{Z}^2 \end{aligned}$$

is $x^* = (1.5, 2.5)$. Obtain a Gomory cut, which cuts of x^* .

Exercise 4.4

We have a budget of 14.000.000 CHF and may invest into the following 4 projects

project	investment	net profit
1	7.000.000 CHF	4.101.000 CHF
2	5.000.000 CHF	3.430.000 CHF
3	4.000.000 CHF	2.005.000 CHF
4	3.000.000 CHF	1.210.000 CHF

The aim is to maximize the net profit, whereby the sum of the investments may not exceed our budget. The projects are 'take it or are leave it' decisions, that means we cannot just invest a fraction of the money. Solve this problem with a suitable dynamic programming approach.

Exercise 4.5

Compute the value of an American call option on a stock with current price equal to 100 CHF, strike price equal to 102 CHF and expiration date four weeks from now. The yearly volatility of the logarithm of the stock return is $\sigma = 0.30$. The risk-free interest rate is 4%. Use a binomial lattice with $N = 4$.