Prof. Friedrich Eisenbrand
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Question session: 28.10.09
Discussion: 4.11.09

Exercises

Optimization Methods in Finance

Fall 2009

Sheet 3

Note: This is just one way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 3.1
Suppose we are given assets $i = 0, \ldots, n$ which are currently (at time 0) priced at S_0^i. There are scenarios ω_j for $j = 1, \ldots, m$, in scenario ω_j asset i will have a price of $S_1^i(\omega_j)$ at time 1. Give an LP, for which any optimum solution gives a portfolio x that provides type-B arbitrage (if such an arbitrage exists).
Hint: Recall that an optimum solution to

$$\begin{aligned} \min \sum_{i=0}^{n} S_0^i \cdot x_i & & \\ \sum_{i=0}^{n} S_1^i(\omega_j) \cdot x_i &\geq 0 & \forall j = 1, \ldots, m \\ x_i &\in \mathbb{R} & \forall i = 1, \ldots, n \end{aligned}$$

is used to detect type-A arbitrage.

Solution:
By definition, a portfolio x provides type-B arbitrage iff we have a non-negative ingoing cash-flow at time 0 and a non-negative ingoing cash flow at time 1 for any scenario, but for at least one scenario we have a strictly positive ingoing cashflow at time 1. Consider the following LP

$$\begin{aligned} \max \sum_{j=1}^{m} y_j & & \\ \sum_{i=0}^{n} S_0^i \cdot x_i &\leq 0 & \\ \sum_{i=0}^{n} S_1^i(\omega_j) \cdot x_i &\geq y_j & \forall j = 1, \ldots, m \\ y_j &\geq 0 & \forall j = 1, \ldots, m \\ x_i &\in \mathbb{R} & \forall i = 1, \ldots, n \end{aligned}$$

Clearly in an optimum solution one has $\sum_{i=0}^{n} S_1^i(\omega_j) = y_j$ (otherwise the y_j's could be increased). In other words, y_j gives the profit in scenario ω_j. If $\sum_{j=1}^{m} y_j > 0$ then there must be at least one j^* with $y_{j^*} > 0$ (and $y_j \geq 0$ for all other j). Vice versa a portfolio with type-B arbitrage yields a feasible solution with positive objective function value.

Exercise 3.2
Consider the Mean Variance Optimization problem

$$\begin{aligned} \max \mu^T x & & \\ x^T Q x &\leq \sigma^2 \\ \sum_{i=1}^n x_i &= 1 \\ x &\geq \mathbf{0} \end{aligned}$$

where μ_i gives the expected return of asset i and Q is the covariance matrix. σ^2 is a given parameter, upper-bounding the variance. x_i gives the ratio, which we are going to invest into asset i.
Suppose we already have a portfolio y (i.e. $y \in \mathbb{R}^n_+$ and $\sum_{i=1}^n y_i = 1$). Increasing the ratio y_i, invested into asset i by some arbitrary $\delta \in [0,1]$, costs $\delta \cdot c_i^+ \geq 0$, whereby decreasing this ratio by δ costs $\delta \cdot c_i^- \geq 0$.
Extend the above Mean Variance Optimization problem, such that the expected return minus the arising transaction costs is maximized (this has to be modeled with linear inequalities/equations). Explain the meaning of newly introduced decision variables.

Solution:
Introduce δ_i^+ as a variable, defining the increase of portfolio i and δ_i^- the decrease of portfolio i.

$$\begin{aligned} \max \mu^T x - \sum_{i=1}^n c_i^+ \delta_i^+ - \sum_{i=1}^n c_i^- \delta_i^- & & \\ x^T Q x &\leq \sigma^2 \\ \sum_{i=1}^n x_i &= 1 \\ \delta_i^+ &\geq x_i - y_i \quad \forall i = 1,\dots,n \\ \delta_i^- &\geq y_i - x_i \quad \forall i = 1,\dots,n \\ x &\geq \mathbf{0} \\ \delta_i^+, \delta_i^- &\geq 0 \quad \forall i = 1,\dots,n \end{aligned}$$

Since $c_i^+, c_i^- \geq 0$ in an optimum solution we would have $\delta_i^+ = \max\{x_i - y_i, 0\}$ and $\delta_i^- = \max\{y_i - x_i, 0\}$, thus the program is correct.

Exercise 3.3
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $x, y \in \mathbb{R}^n$. Prove that $g : [0,1] \to \mathbb{R}$ with $g(t) = f(tx + (1-t)y)$ is convex as well.

Solution:

Let $t_1, t_2, \lambda \in [0,1]$. Then

$$\begin{array}{rcl} g(\lambda t_1 + (1-\lambda)t_2) & = & f((\lambda t_1 + (1-\lambda)t_2)\cdot x + (1-(\lambda t_1 + (1-\lambda)t_2))\cdot y) \\ & = & f((\lambda t_1 + (1-\lambda)t_2)\cdot x + (\lambda(1-t_1) + (1-\lambda)(1-t_2))\cdot y) \\ & = & f(\lambda(t_1 x + (1-t_1)y) + (1-\lambda)\cdot(t_2 x + (1-t_2)y)) \\ & \overset{f \text{ convex}}{\leq} & \lambda \cdot f(t_1 x + (1-t_1)y) + (1-\lambda)\cdot f(t_2 x + (1-t_2)y) \\ & = & \lambda \cdot g(t_1) + (1-\lambda)\cdot g(t_2) \end{array}$$

Exercise 3.4

Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Show that Q is positive semidefinite (i.e. $\forall x \in \mathbb{R}^n : x^T Q x \geq 0$) if and only if all eigenvalues of Q are non-negative.

Hint: You may use the following theorem from linear algebra: *Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, there are eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with eigenvectors $v_1, \ldots, v_n \in \mathbb{R}^n$ (i.e. $Av_i = \lambda_i v_i$ for $i = 1, \ldots, n$), which form an orthonormal basis of the $\mathbb{R}^n$ (that means $v_i v_j = 0$ for all $i \neq j$ and $v_i v_i = 1$ for all $i = 1, \ldots, n$).*

Solution:

Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be the eigenvalues with orthonormal eigenvectors $v_1, \ldots, v_n$ according to the above theorem.

$\Rightarrow$. Suppose there is an $i \in \{1, \ldots, n\}$ with $\lambda_i < 0$, then $v_i^T Q v_i = v_i^T(\lambda_i v_i) = \underbrace{\lambda_i}_{<0}\underbrace{v_i^T v_i}_{=1} < 0$ thus Q is not positive-semidefinite.

$\Leftarrow$. Suppose $\lambda_1, \ldots \lambda_n \geq 0$. Let $x \in \mathbb{R}^n$. Since $v_1, \ldots, v_n$ are a basis, we can write

$$x = \mu_1 v_1 + \ldots + \mu_n v_n$$

But then

$$\begin{array}{rcl} x^T Q x & = & (\mu_1 v_1 + \ldots + \mu_n v_n)^T Q(\mu_1 v_1 + \ldots + \mu_n v_n) \\ & = & \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i \mu_j v_i^T \underbrace{(Q v_j)}_{=\lambda_j v_j} \\ & = & \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i \mu_j \lambda_j \cdot \underbrace{v_i^T v_j}_{=0 \text{ if } i\neq j \text{ and } 1 \text{ otherwise}} \\ & = & \sum_{i=1}^{n} \underbrace{\mu_i^2}_{\geq 0}\underbrace{\lambda_j}_{\geq 0} \\ & \geq & 0 \end{array}$$

using that $v_i \perp v_j$ for $i \neq j$.
