Prof. Friedrich Eisenbrand — Question session: 28.10.09
Location: MA A3 31 — Discussion: 4.11.09

Exercises

Optimization Methods in Finance

Fall 2009

Sheet 3

Exercise 3.1
Suppose we are given assets $i = 0, \dots, n$ which are currently (at time 0) priced at S_0^i. There are scenarios ω_j for $j = 1, \dots, m$, in scenario ω_j asset i will have a price of $S_1^i(\omega_j)$ at time 1. Give an LP, for which any optimum solution gives a portfolio x that provides type-B arbitrage (if such an arbitrage exists).
Hint: Recall that an optimum solution to

$$\begin{aligned} \min \sum_{i=0}^{n} S_0^i \cdot x_i \\ \sum_{i=0}^{n} S_1^i(\omega_j) \cdot x_i &\geq 0 \quad \forall j = 1, \dots, m \\ x_i &\in \mathbb{R} \quad \forall i = 0, \dots, n \end{aligned}$$

is used to detect type-A arbitrage.

Exercise 3.2
Consider the Mean Variance Optimization problem

$$\begin{aligned} \max \mu^T x \\ x^T Q x &\leq \sigma^2 \\ \sum_{i=1}^{n} x_i &= 1 \\ x &\geq \mathbf{0} \end{aligned}$$

where μ_i gives the expected return of asset i and Q is the covariance matrix. σ^2 is a given parameter, upper-bounding the variance. x_i gives the ratio, which we are going to invest into asset i.
Suppose we already have a portfolio y (i.e. $y \in \mathbb{R}^n_+$ and $\sum_{i=1}^n y_i = 1$). Increasing the ratio y_i, invested into asset i by some arbitrary $\delta \in [0,1]$, costs $\delta \cdot c_i^+ \geq 0$, whereby decreasing this ratio by δ costs $\delta \cdot c_i^- \geq 0$.
Extend the above Mean Variance Optimization problem, such that the expected return minus the arising transaction costs is maximized (this has to be modeled with linear inequalities/equations). Explain the meaning of newly introduced decision variables.

Exercise 3.3

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $x, y \in \mathbb{R}^n$. Prove that $g : [0,1] \to \mathbb{R}$ with $g(t) = f(tx + (1-t)y)$ is convex as well.

Exercise 3.4

Let $Q \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Show that Q is positive semidefinite (i.e. $\forall x \in \mathbb{R}^n : x^T Q x \geq 0$) if and only if all eigenvalues of Q are non-negative.

Hint: You may use the following theorem from linear algebra: *Given a symmetric matrix $A \in \mathbb{R}^{n \times n}$, there are eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ with eigenvectors $v_1, \dots, v_n \in \mathbb{R}^n$ (i.e. $Av_i = \lambda_i v_i$ for $i = 1, \dots, n$), which form an orthonormal basis of the $\mathbb{R}^n$ (that means $v_i v_j = 0$ for all $i \neq j$ and $v_i v_i = 1$ for all $i = 1, \dots, n$).*