Exercises

Optimization in Finance

Fall 2008

Sheet 3

Exercise 3.1
Define

$$Q = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \text{ and } C = \{x \in \mathbb{R}^2 \mid x^T Q x \leq 19\}$$

Clearly $x^* = (2,1) \notin C$. Compute a hyperplane separating x^* from C.
Hint: Recall the gradient.

Exercise 3.2
Proof the following

1. Given a closed, convex set $S \subseteq \mathbb{R}^n$ and a point $x^* \in \mathbb{R}^n$ with $x^* \notin S$. Show that there is a hyperplane $c^T x = \delta$ with $c^T s < \delta$ for any $s \in S$ and $c^T x^* > \delta$ (this is Theorem 3.8 without boundedness of S).

2. Let $S, R \subseteq \mathbb{R}^n$ be disjoint compact convex sets. Show that there is a hyperplane $c^T x = \delta$ with $c^T s < \delta$ and $c^T r > \delta$ for any $s \in S$ and $r \in R$.

Hint: You may use the argument from the lecture, that whenever $s^* \in S, r^* \in R$ attain the minimum distance of 2 convex disjoint sets S, R, then the hyperplane $c^T x = \delta$ through $\frac{s^*+r^*}{2}$, orthogonal to $s^* - r^*$ strictly separates S and R.

Exercise 3.3
Suppose you are given an oracle algorithm, which for a given polyhedron $P = \{x \mid Ax \leq b\}$ gives you a feasible solution or asserts that there is none. Show that using a <u>single</u> call of this oracle one can obtain an <u>optimum</u> solution for the LP

$$\min\{Ax = b;\ x \geq \mathbf{0}\},$$

assuming that the LP is feasible and bounded.
Hint: Use duality.

Theorem. (Strong Separation Theorem) *Let C_1, C_2 be convex sets, such that C_1 is closed. Then there is a hyperplane $c^T x = \delta$, such that $c^T x \leq \delta$ for $x \in C_1$ and $c^T x > \delta$ for $x \in C_2$.*

Exercise 3.4 - Farkas Lemma
Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. We want to show that exactly one of the following 2 properties hold:

1. $\exists x \geq \mathbf{0} : Ax \leq b$
2. $\exists y \geq \mathbf{0} : y^T A \geq \mathbf{0},\ y^T b < 0$

Do the following:

1. Show that it is impossible that both properties hold together.
2. Suppose (1) does not hold. Show that then (2) must hold.
   Hint: Let $a_1, \ldots, a_m$ be the columns of A. Then falseness of property (1) is equivalent to

   $$b \notin \text{cone}\{a_1, \ldots, a_m, e_1, \ldots, e_n\} = \{\lambda_1 a_1 + \ldots + \lambda_m a_m + \mu_1 e_1 + \ldots + \mu_n e_n \mid \lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n \geq 0\}$$

   Recall that cones are convex and apply the above Strong Separation Theorem.

Exercise 3.5 - Another proof for strong duality
Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. Suppose that the sets $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\ x \geq \mathbf{0}\}$ and $D = \{y \in \mathbb{R}^m \mid y^T A \geq c^T,\ y \geq \mathbf{0}\}$ are nonempty. Show that the system

$$\begin{aligned} Ax &\leq b \\ y^T A &\geq c^T \\ c^T x &\geq y^T b \\ x &\geq \mathbf{0} \\ y &\geq \mathbf{0} \end{aligned}$$

has a solution. Here you may use weak duality but not the strong duality theorem.
Hint: Choose $\delta := \max\{c^T x \mid Ax \leq b, x \geq \mathbf{0}\}$. Then apply Farkas Lemma to the system $Ax \leq b,\ x \geq \mathbf{0},\ c^T x \geq \delta + \varepsilon$ for $\varepsilon > 0$.

Exercise 3.6
In a combinatorial exchange, both buyers and sellers can submit combinatorial bids. Bids are like in the multiple item case of combinatorial auctions, except that the λ_i^j values can be negative, as the prices p_j, representing selling instead of buying. Note that a single bid can be buying some items, while selling other items. Write an integer program that will maximize the surplus by the combinatorial exchange.