Prof. Friedrich Eisenbrand
Location: MA A3 31

Question session: 14.10.09
Discussion: 21.10.09

Exercises

Optimization Methods in Finance

Fall 2009

Sheet 2

Note: This is just one way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 2.1
We know that the dual of

$$(P) \quad \min\{c^T x \mid Ax = b;\ x \geq \mathbf{0}\}$$

is

$$(D) \quad \max\{b^T y \mid A^T y \leq c\}.$$

Use this fact to obtain the dual LP for

$$\min\{c^T x \mid Ax \geq b,\ x \geq \mathbf{0}\}.$$

Solution:
We know that

$$\begin{aligned}
& \min\{c^T x \mid Ax \geq b,\ x \geq 0\} \\
= \quad & \min\{(c,\mathbf{0})^T (x,z) \mid (A,-I)\begin{pmatrix} x \\ z \end{pmatrix} = b; (x,z) \geq \mathbf{0}\} \\
\overset{duality}{=} \quad & \max\{b^T y \mid (A,-I)^T y \leq (c,\mathbf{0})\} \\
= \quad & \max\{b^T y \mid A^T y \leq c;\ -I^T y \leq \mathbf{0}\} \\
= \quad & \max\{b^T y \mid A^T y \leq c;\ y \geq \mathbf{0}\}
\end{aligned}$$

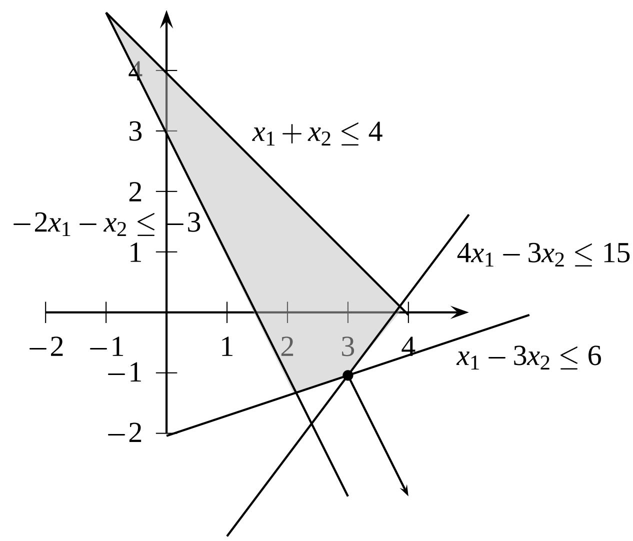
Given that both systems are feasible.

Exercise 2.2
Consider the linear program

$$\begin{array}{rcl} \max\ x_1 - 2x_2 & & (P) \\ x_1 + x_2 & \leq & 4 \\ x_1 - 3x_2 & \leq & 6 \\ -2x_1 - x_2 & \leq & -3 \\ 4x_1 - 3x_2 & \leq & 15 \end{array}$$

1. State the dual program (to (P)).
2. The vector $x^* = (x_1^*, x_2^*) = (3, -1)$ is a unique optimal solution for (P) (you do not have to show that). Use this to obtain an optimum solution for the dual and argue why this solution is optimal for the dual.
   Hint: Use complementary slackness.

Solution:
To check whether $x^* = (3, -1)$ is really optimal, consider



For (1) The dual is

$$\begin{array}{lccccccc} \min & 4y_1 & +6y_2 & -3y_3 & +15y_4 & & \\ & y_1 & +y_2 & -2y_3 & +4y_4 & = & 1 \\ & y_1 & -3y_2 & -y_3 & -3y_4 & = & -2 \\ & y_1, & y_2, & y_3, & y_4, & \geq & 0 \end{array}$$

For (2). Let $y \in \mathbb{R}^4_+$ be an optimum dual solution. Only the 2nd and 4th primal inequality are tight for x^*, thus we know that $y_1 = y_3 = 0$ by complementary slackness. We solve the equation

$$\begin{bmatrix} y_2 & +4y_4 & = & 1 \\ -3y_2 & -3y_4 & = & -2 \end{bmatrix} \Rightarrow y_2 = 5/9, y_4 = 1/9$$

Due to complementary slackness $y = (0, 5/9, 0, 1/9)$ must be an optimal dual solution. (Alternatively: Compare the objective function values $3 - 2 \cdot (-1) = 5 = \frac{30+15}{9} = (4, 6, -3, 15) \cdot (0, 5/9, 0, 1/9)^T$)

Exercise 2.3

Suppose we have the following European Call options, all w.r.t. the same underlying asset (and maturity) which is currently priced at 40 CHF:

Option i	strike price K_i (in CHF)	price S_0^i (in CHF)
1	30	10
2	40	7
3	50	10/3
4	60	0

Construct a portfolio of the above options that provides a type-A arbitrage opportunity.
Hint: You may use any LP solver.

Solution:

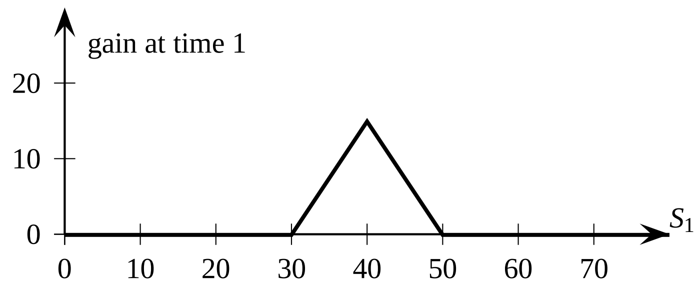
Recall that a European Call option with price p and strike price c means that we can buy for a price of p at time 0 the right to buy the underlying asset for a price c at time 1. Let x_i be the amount of options i that we buy ($x_i < 0$ means we sell $|x_i|$ times option i). We can detect a type-A arbitrage using the following LP

$$
\begin{aligned}
\min \sum_{i=1}^{4} S_0^i x_i \\
\sum_{i=1}^{4} \max\{K_1 - K_i, 0\} x_i &\geq 0 \\
\sum_{i=1}^{4} \max\{K_2 - K_i, 0\} x_i &\geq 0 \\
\sum_{i=1}^{4} \max\{K_3 - K_i, 0\} x_i &\geq 0 \\
\sum_{i=1}^{4} \max\{K_4 - K_i, 0\} x_i &\geq 0 \\
\sum_{i=1}^{4} (\max\{K_4 - K_i + 1, 0\} - \max\{K_4 - K_i, 0\}) x_i &\geq 0 \\
x_i &\in \mathbb{R}
\end{aligned}
$$

which is

$$
\begin{array}{lccccll}
\min & 10x_1 & +7x_2 & +\frac{10}{3}x_3 & +0x_4 & & \\
 & 10x_1 & & & & \geq & 0 \\
 & 20x_1 & +10x_2 & & & \geq & 0 \\
 & 30x_1 & +20x_2 & +10x_3 & & \geq & 0 \\
 & x_1 & +x_2 & +x_3 & +x_4 & \geq & 0 \\
 & x_1, & x_2, & x_3, & x_4 & \in & \mathbb{R}
\end{array}
$$

(and $10x_1 + 7x_2 + \frac{10}{3}x_3 + 0x_4 = -1$ for normalization). We obtain a (not unique) solution $x = (1.5, -3, 1.5, 0)$ giving a negative objective function value (namely -1). Depending on the price S_1 of the underlying asset at time 1 we furthermore earn the following amount at time 1 (additionally to the 1 CHF that we got at time 0):

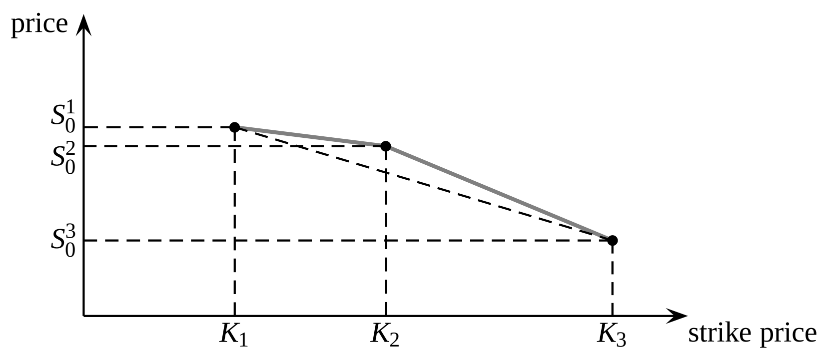


Exercise 2.4
Suppose we are given 3 European Call options (all w.r.t. the same underlying asset, all with the same maturity), Option i with a price of S_0^i and strike price of K_i. Suppose that $K_1 < K_2 < K_3; S_0^1 > S_0^2 > S_0^3$ and the point (K_2, S_0^2) lies above (or on) the line segment that connects (K_1, S_0^1) and (K_3, S_0^3). Formally there is a $0 < \lambda < 1$ with $K_2 = \lambda K_1 + (1-\lambda)K_3$ and

$$S_0^2 \geq \lambda S_0^1 + (1-\lambda)S_0^3.$$

Give an *explicit* formula for a portfolio that provides arbitrage. Which type of arbitrage is it?

Solution:
The situation can be depicted as follows:

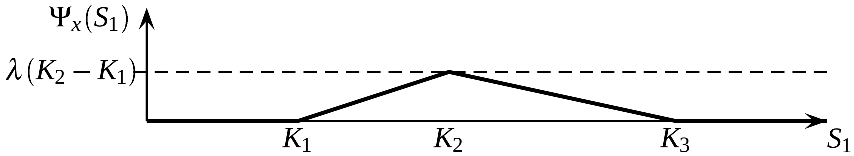


We choose a portfolio $x \in \mathbb{R}^3$ with $x_1 = \lambda, x_2 = -1, x_3 = (1-\lambda)$. Then $\sum_{i=1}^3 S_0^i x_i = S_0^1\lambda - S_0^2 + (1-\lambda)S_0^3 \leq 0$ by assumption, hence we have a non-negative ingoing cash-flow at time 0. On the other hand, let us consider the gain at time 1

$$\Psi_x(S_1) = \sum_{i=1}^{3} \max\{S_1 - K_i, 0\} \cdot x_i$$

depending on the price S_1 which the asset reaches. We verify that $\forall S_1 \geq 0 : \Psi_x(S_1) \geq 0$ and $\exists S_1 \geq 0 : \Psi_x(S_1) > 0$:

- $S_1 = K_1 : \Psi_x(K_1) = 0$
- $S_1 = K_2 : \Psi_x(K_2) = \lambda(K_2 - K_1) > 0$
- $S_1 = K_3 : \Psi_x(K_3) = \lambda(K_3 - K_1) - (K_3 - K_2) = -(\lambda K_1 + (1-\lambda)K_3) + K_2 = 0$
- $S_1 \to \infty : \Psi_x(K_3 + 1) - \Psi_x(K_3) = x_1 + x_2 + x_3 = 0$



In other words, the payoff at time 1 is never negative and for $S_1 \in]K_1, K_3[$ it is strictly positive. Hence the portfolio x provides a type-B arbitrage. If the point (K_2, S_0^2) lies *strictly* above the line segment connecting (K_1, S_0^1) and (K_3, S_0^3), then x additionally provides type-A arbitrage since then $S_0^1\lambda - S_0^2 + (1-\lambda)S_0^3 < 0$, hence the ingoing cash flow at time 0 would be strictly positive.