Prof. Friedrich Eisenbrand
Location: MA A3 31

Question session: 30.09.09
Discussion: 7.10.09

Exercises

Optimization Methods in Finance

Fall 2009

Sheet 1

Note: This is just <u>one</u> way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 1.1

Write the following linear program in standard form

$$\begin{aligned} \max 4x_1 + x_2 - x_3 & \\ x_1 + 3x_3 &\leq 6 \\ 3x_1 + x_2 + 3x_3 &\geq 9 \\ x_1, x_2 &\geq 0 \\ x_3 &\in \mathbb{R} \end{aligned}$$

Solution:

We replace $x_3 \in \mathbb{R}$ by $x_3 = x_3^+ - x_3^-$ with $x_3^+, x_3^- \geq 0$ and add slackvariables $s_1, s_2 \geq 0$. Then the LP is

$$\begin{array}{lrrrrrrcl} \min & -4x_1 & -x_2 & +x_3^+ & -x_3^- & & & & \\ & x_1 & & +3x_3^+ & -3x_3^- & +s_1 & & = & 6 \\ & 3x_1 & +x_2 & +3x_3^+ & -3x_3^- & & -s_2 & = & 9 \\ & x_1, & x_2, & x_3^+, & x_3^-, & s_1, & s_2 & \geq & 0 \end{array}$$

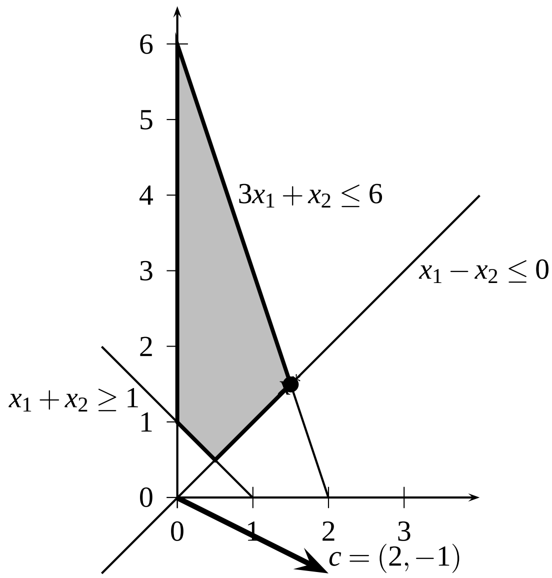
Exercise 1.2

Draw the feasible region (set of feasible solutions) of the following linear program (with 2 variables)

$$\begin{aligned} \max 2x_1 - x_2 & \\ x_1 + x_2 &\geq 1 \\ x_1 - x_2 &\leq 0 \\ 3x_1 + x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Determine the optimal solution to this problem by inspecting your drawing.

Solution:



Optimum solution x^* solves

$$\left| \begin{array}{cccc} x_1 & -x_2 & = & 0 \\ 3x_1 & +x_2 & = & 6 \end{array} \right| \Leftrightarrow (x_1, x_2) = (1.5, 1.5)$$

Exercise 1.3

i) For

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & -1 & 4 \end{pmatrix}$$

compute a $d \in \mathbb{R}^3, d \neq \mathbf{0}$ with $Ad = \mathbf{0}$.

ii) Invert the matrix

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 1 \end{pmatrix}$$

iii) Compute a solution $x \in \mathbb{R}^3$ for $Bx = b$ with $b = (3, 1, -1)^T$.

Solution:

i) We apply elementary row operations:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & -1 & 4 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right)$$

hence $d = (-1, 1, 1)^T$ is a non-trivial solution.

ii) Again we apply elementary row operations to turn the left hand side matrix into a unit matrix

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & -3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

Hence

$$B^{-1} = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

iii) The unique solution is

$$x = B^{-1}b = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \\ 0 \end{pmatrix}$$

Exercise 1.4

The vector $x^* = (0,1,1,1)$ is an optimal solution of

$$\begin{aligned} \min\ (1,1,0,2) \cdot x \\ \underbrace{\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix}}_{=A} x &= \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} \\ x &\geq \mathbf{0} \end{aligned}$$

with $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Use the proof of Lemma 2.1 to find another optimal solution x' such that $A_{J'}$ has full column rank with $J' = \{i \mid x'_i > 0\}$.

Solution:

Here $J = \{i \mid x^*_i > 0\} = \{2,3,4\}$ thus $A_J = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ does not have full column rank. Note that

$$\text{kern} \begin{pmatrix} A \\ e_1^T \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 0 \\ -4 \\ -1 \\ 2 \end{pmatrix} \right\}$$

Thus choose $d = (0,-4,-1,2)$ then $Ad = \mathbf{0}$ and $d(j) = 0\, \forall j \notin J$. (Observe that $c^T d = (1,1,0,2) \cdot (0,-4,-1,2)^T = 0$). Choose ε maximal, such that

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ -4 \\ -1 \\ 2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $\varepsilon = 1/4$. Then

$$x' := x + \varepsilon d = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 \\ -4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3/4 \\ 3/2 \end{pmatrix}$$

Now $J' = \{3,4\}$ thus

$$A_J = \begin{pmatrix} 2 & 3 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

which has full column rank.

Exercise 1.5 – Practical exercise (2 points)
For the first practical exercise do the following (see the lecture notes for more details):

1. Transform the cashflow LP from the lecture into standard form.
2. Implement the naive linear programming algorithm to find an optimum solution (you can use the Boost C++-library).
3. Send the code together with compile instructions to `thomas.rothvoss@epfl.ch` until the 30th of September.