Exercises

Optimization in Finance

Fall 2008

Sheet 1

Exercise 1.1
Write the following linear program in standard form

$$\begin{aligned} & \max 4x_1 + x_2 - x_3 \\ x_1 + 3x_3 &\leq 6 \\ 3x_1 + x_2 + 3x_3 &\geq 9 \\ x_1, x_2 &\geq 0 \\ x_3 &\in \mathbb{R} \end{aligned}$$

Exercise 1.2
Draw the feasible region (set of feasible solutions) of the following linear program (with 2 variables)

$$\begin{aligned} & \max 2x_1 - x_2 \\ x_1 + x_2 &\geq 1 \\ x_1 - x_2 &\leq 0 \\ 3x_1 + x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Determine the optimal solution to this problem by inspecting your drawing.

Exercise 1.3
Consider the following linear program

$$\begin{aligned} & \min 2x_1 + 3x_2 \\ x_1 + x_2 &\geq 5 \\ x_1 &\geq 1 \\ x_2 &\geq 2 \end{aligned}$$

Prove that $x^* = (3,2)$ is the optimal solution by showing that the objective value of any feasible solution is at least 12 (hint: duality).

Exercise 1.4
A polyhedron is a set $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$. Show that each polyhedron is convex (recall that a set $C \subseteq \mathbb{R}^n$ is convex if and only if for any $x, y \in C$ one has $\lambda x + (1-\lambda)y \in C$ for all $\lambda \in [0,1]$).

Exercise 1.5
The vector $x^* = (0,1,1,1)$ is an optimal solution of

$$\begin{aligned} &\min\,(1,1,0,2)\cdot x \\ &\underbrace{\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix}}_{=A} x \;=\; \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} \\ &\qquad\qquad\quad x \;\geq\; \mathbf{0} \end{aligned}$$

with $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Use the proof of Lemma 1.1 to find another optimal solution x' such that $A_{J'}$ has full column rank with $J' = \{i \mid x'_i > 0\}$.

Exercise 1.6
For our company we expect the following net cash flow

Quarter	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
net cash flow	-100	-500	-100	600	500	-200	-600	900

(meaning that for instance we have to pay 100 units at the beginning of quarter 1, while we get 600 units at beginning of quarter 4). Initially we have 0 units. There are a couple of options how to invest/borrow money:

- A 2-year loan (to borrow money) available only at the beginning of Q1 (has to be payed back at beginning of Q9) for 8% interest (for 2 years).
- A 6-month loan available each quarter with an interest of 3.6% (for 6 months)
- A 3-month loan available each quarter at an interest of 2.5% (per quarter)
- At any beginning of a quarter we can invest money for 0.5% per quarter.

Ensure that at Q9, we have payed back all loans. We want to maximize our wealth at the beginning of Q9.

Formulate this cash-flow management problem as a linear program (introduce suitable decision variables). Then model the LP with `ZIMPL` and solve it with `QSOpt`. Interpret the outcome.