Prof. Friedrich Eisenbrand
Location: MA A3 31

Question session: 10.11.10
Discussion: 17.11.10

Exercises

Optimization Methods in Finance

Fall 2010

Sheet 4

Exercises marked with **(*)** qualify for one bonus point, if correctly presented in the discussion session

Exercise 4.1 (*)
Consider the primal

$$\max\{c^T x \mid Ax \leq b\} \quad (P)$$

and the dual LP.

$$\min\{y^T b \mid y^T A = c^T, y \geq \mathbf{0}\} \quad (D)$$

i) Suppose that (P) is feasible and bounded, say $x^* \in \mathbb{R}^n$ is an optimal solution. Let $I \subseteq \{1,\dots,m\}$ be the set of active constraints at x^* (i.e. $I = \{i \in \{1,\dots,m\} \mid A_i x^* = b_i\}$ and A_i denotes the ith row of A). Show that there exists a $y^* \in \mathbb{R}^m$ with

$$y_i^* \geq 0 \;\forall i \in I, \quad y_i^* = 0 \;\forall i \notin I, \quad {y^*}^T A = c^T$$

Hint: Assume for contradiction that there is no such y^*, i.e. $c \notin \{\sum_{i \in I} A_i y_i \mid z_i \geq 0\}$ and apply the strict separating hyperplane theorem: *Given a closed convex set C and a point $x_0 \notin C$, there exists a hyperplane $a^T x = \beta$ with $a^T x_0 < \beta$, $a^T x > \beta \;\forall x \in C$.* Then show that x^* would not be optimal.

ii) Show that the vector y^* from $i)$ is an optimal dual solution with objective function value $c^T x^*$.

iii) Suppose that (P) is infeasible and the dual problem (D) is feasible. Show that the dual problem is unbounded.
Hint: Show that there is a $v \in \mathbb{R}^m \backslash \{\mathbf{0}\}$ with $A^T v = 0, v \geq \mathbf{0}, b^T v < 0$.

Exercise 4.2 (*)
Let x^* be a solution to

$$\min\{c^T x \mid Ax = b, x \geq \mathbf{0}\} \quad (P)$$

and y^* be a feasible solution to

$$\max\{b^T y \mid A^T y \leq c\} \quad (D)$$

Prove that the following conditions are equivalent

1. x^* and y^* are both optimal (i.e. x^* optimal for (P) and y^* optimal for (D))
2. $\forall i : x_i^* > 0 \Rightarrow (c - A^T y^*)_i = 0$

Hint: Recall that by strong duality, the optimal values for (P) and (D) are the same, given that both systems are feasible.

Exercise 4.3 (*)
Suppose we have the following European Call options, all w.r.t. the same underlying asset (and maturity) which is currently priced at 40 CHF:

Option i	strike price K_i (in CHF)	price S_0^i (in CHF)
1	30	10
2	40	7
3	50	10/3
4	60	0

Construct a portfolio of the above options that provides a type-A arbitrage opportunity.
Hint: You may use any LP solver.

Exercise 4.4 (*)
Suppose we are given 3 European Call options (all w.r.t. the same underlying asset, all with the same maturity), Option i with a price of S_0^i and strike price of K_i. Suppose that $K_1 < K_2 < K_3; S_0^1 > S_0^2 > S_0^3$ and the point (K_2, S_0^2) lies above (or on) the line segment that connects (K_1, S_0^1) and (K_3, S_0^3). Formally there is a $0 < \lambda < 1$ with $K_2 = \lambda K_1 + (1-\lambda)K_3$ and

$$S_0^2 \geq \lambda S_0^1 + (1-\lambda)S_0^3.$$

Give an *explicit* formula for a portfolio that provides arbitrage. Which type of arbitrage is it?