Prof. Friedrich Eisenbrand — Question session: 27.10.10
Location: MA A3 31 — Discussion: 03.11.10

Exercises

Optimization Methods in Finance

Fall 2010

Sheet 3

Exercises marked with **(*)** qualify for one bonus point, if correctly presented in the discussion session

Exercise 3.1 (*)
Consider the optimization problem

$$\begin{aligned} \min x^2+1 & \\ (x-2)(x-4) & \leq 0 \\ x & \in \mathbb{R} \end{aligned}$$

i) *Analysis of primal problem.* Give the feasible set, the optimal value and the optimal solution.

ii) *Lagrangian and dual function.* Plot the function x^2+1 versus x. One the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x,\lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \geq \inf_x L(x,\lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

iii) *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimum solution λ^*. Does strong duality hold?

Exercise 3.2 (*)
In this exercise, we want to show an example of a convex program, where strong duality fails. Consider the optimization problem

$$\begin{aligned} \min e^{-x} & \\ x^2/y & \leq 0 \\ (x,y) & \in D \end{aligned}$$

with $D := \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$.

i) Verify that this is a convex optimization problem. Find the optimal value.

ii) Give the Lagrange dual problem, and find the optimal solution λ^* and optimum value d^* of the dual program. What is the optimal duality gap?

iii) Does Slater's condition hold for this problem?

Exercise 3.3 (*)
In this exercise, we want to argue, why the RWMA (which can minimize convex functions over the simplex $\Sigma^m := \text{conv}\{e_1,\ldots,e_m\} = \{\lambda \in \mathbb{R}^m \mid \sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0\})$ can also be used to optimize over general polytopes. Here, we are motivated, since the minimum variance portfolio problem is a convex optimization problem over the domain $\{x \in \mathbb{R}^N \mid \sum_{i=1}^N x_i = 1, \sum_{i=1}^N \overline{r}_i x_i \geq r, x \geq 0\}$ which is Σ^N intersected with the half-space $\sum_{i=1}^N \overline{r}_i x_i \geq r$.
Let $v_1,\ldots,v_m \in \mathbb{R}^n$ and let $Q := \text{conv}\{v_1,\ldots,v_m\} := \{\sum_{i=1}^m \lambda_i v_i \mid \sum_{i=1}^m \lambda_i = 1, \lambda_1,\ldots,\lambda_m \geq 0\}$. Define $q : \Sigma^m \to Q$ with $q(\lambda) = \sum_{i=1}^m \lambda_i x_i$. Let $f : Q \to \mathbb{R}$ be a convex function. Show that

i) The function $g : \Sigma^m \to \mathbb{R}$ with $g(\lambda) = f(q(\lambda))$ is convex.

ii) One has
$$\min_{\lambda \in \Sigma^m} g(\lambda) = \min_{x \in Q} f(x)$$

iii) Describe $\sum^N \cap \{x \in \mathbb{R}^N \mid \sum_{i=1}^N \overline{r}_i x_i \geq r\}$ as the convex hull of at most $N^2 + N$ points and conclude that the RWMA can be used to solve the portfolio optimization problem $\min\{x^T Q x \mid x \in \sum^N \cap \{x \in \mathbb{R}^N \mid \sum_{i=1}^N \overline{r}_i x_i \geq r\}\}$.

Exercise 3.4 (*)
Let $D \subseteq \mathbb{R}^n$ be a convex set and $f_0,\ldots,f_m : D \to \mathbb{R}$ be convex functions. Show that the set

$$A = \{(u,t) \in \mathbb{R}^m \times \mathbb{R} \mid \exists x \in D : f_i(x) \leq u_i,\ f_0(x) \leq t\}$$

is convex.

Exercise 3.5 (*)
Let $f : D \longrightarrow \mathbb{R}$ be a convex function for some convex domain $D \subseteq \mathbb{R}^n$. Show that

i) The function $f(x)^2$ is convex, given that $f(x) \geq 0$ for all $x \in D$.

ii) $f(Ax+b)$ is convex for any $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

Conclude that a function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = x^T \cdot Q \cdot x$ and $Q \in \mathbb{R}^{n\times n}$, $Q \succeq 0$ is convex.