Prof. Friedrich Eisenbrand
Location: MA A3 31

Question session: 30.09.10
Discussion: 07.10.10

Exercises

Optimization Methods in Finance

Fall 2010

Sheet 1

Exercises marked with **(*)** qualify for one bonus point, if correctly presented in the discussion session

Exercise 1.1
Let X be a finite set of N elements (assume that N is a power of 2). At least one of these elements is *interesting*. You may ask questions of the form: *Does $H \subseteq X$ contain an interesting element?* Design an algorithm that identifies an interesting element by asking at most $\log_2(N)$ questions of this kind.

Can there exist a deterministic algorithm that identifies an interesting element by asking fewer questions in the worst case?

Exercise 1.2
Show the following inequalities for $0 \leq \varepsilon \leq 1/2$:

1. $(1-\varepsilon)^x \leq (1-\varepsilon x)$ for $x \in [0,1]$.
2. $(1+\varepsilon)^{-x} \leq (1-\varepsilon x)$ for $x \in [-1,0]$.
3. $\ln\left(\frac{1}{1-\varepsilon}\right) \leq \varepsilon + \varepsilon^2$.
4. $\ln(1+\varepsilon) \geq \varepsilon - \varepsilon^2$.

Exercise 1.3 (*)
Consider the randomized weighted majority algorithm and suppose that the loss-vectors at time t satisfy $\ell^t \in [0,\rho]^N$ for $t = 0,\dots,T$. Show that the expected loss of the forecaster is bounded by

$$E[L] \leq \frac{\rho \cdot \ln N}{\varepsilon} + (1+\varepsilon)\cdot L^j,$$

if one uses the update rule $w_j := w_j(1-\varepsilon)^{\ell_j^T/\rho}$. As in the lecture, $L^j = \sum_{t=0}^T \ell_j^t$ is the loss accumulated by expert j.

Exercise 1.4 (*)
Suppose that the loss vectors satisfy $\ell^t \in [-1,1]^N$ for $t = 0,\dots,T$ and consider the following updating rule

$$w_j^{t+1} := \begin{cases} w_j^t(1-\varepsilon)^{\ell_j^t} & \text{if } \ell_j^t \geq 0 \\ w_j^t(1+\varepsilon)^{-\ell_j^t} & \text{if } \ell_j^t < 0. \end{cases}$$

Show that

$$E[L] \leq \frac{\ln N}{\varepsilon} + (1+\varepsilon) \cdot \sum_{t:\ell_j^t \geq 0} \ell_j^t + (1-\varepsilon) \cdot \sum_{t:\ell_j^t < 0} \ell_j^t$$

holds. Show furthermore that this is also guaranteed if the update rule $w_j^{t+1} := w_j(1-\varepsilon\ell_j^t)$ is used.

Exercise 1.5 (*)

Suppose you have some initial belief about the quality of the experts. This belief is represented by a probability distribution on the experts $p_j,\ j = 1,\ldots,N$ with $p_j > 0$ and $\sum_{j=1}^n p_j = 1$. We modify the weighted majority algorithm by setting the initial weights $w_j := p_j$. Show that this modification results in a guarantee

$$E[L] \leq \frac{\ln(1/p_j)}{\varepsilon} + (1+\varepsilon) \cdot L^j.$$

Suppose now that we have a countably infinite number of experts. Use the result above to argue that one can guarantee

$$E[L] \leq \frac{2 \cdot \ln(j) + 10}{\varepsilon} + (1+\varepsilon) \cdot L^j.$$

by choosing a suitable probability distribution on the experts.