# Integer Points in Polyhedra

Spring 2009

Assignment Sheet 8

**Exercise 1 (Rational generating functions for cones)**
Let $C$ be a cone in $\mathbb{R}^n$ generated by linearly independent vectors $C = \text{cone}(u_1, u_2, \ldots, u_k)$ for some integral vectors $u_1, u_2, \ldots, u_k \in \mathbb{R}^n$. Show that

$$f(C;x) = \left(\prod_{m \in \Pi \cap \mathbb{Z}^n} x^m\right) \prod_{i=1}^{k} \frac{1}{1 - x^{u_i}},$$

where $\Pi$ is the fundamental parallelepiped generated by $u_1, u_2, \ldots, u_k$:

$$\Pi = \left\{ \sum_{i=1}^{k} \lambda_i u_i : 0 \le \lambda_i < 1,\ i = 1, 2, \ldots, k \right\}.$$

Is it also true when $u_1, u_2, \ldots, u_k$ are not linearly independent?

**Exercise 2 (Rational generating functions for not pointed polyhedra)**
Let $P$ be a rational polyhedron containing a straight line. Show that $f(P;x) \equiv 0$.

**Exercise 3 (Brion's theorem)**
Let $P$ be a rational polyhedron. Prove that

$$f(P;x) = \sum_{v \in \text{Vert}(P)} f(\text{cone}(P, v); x).$$

**Exercise 4 (The reciprocity relation)**
Let $C$ be a pointed rational cone in $\mathbb{R}^n$ with non-empty interior $\text{int}(C)$. Prove that

$$f(\text{int}(C); x^{-1}) = (-1)^n f(C;x).$$