Integer Points in Polyhedra

Spring 2009

Assignment Sheet 6

Exercise 1 (Inclusion–exclusion)
Let $A_1, A_2, \dots, A_n \subseteq \mathbb{R}^n$ be sets. Prove the following inclusion–exclusion formula:

$$\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{I} (-1)^{|I|-1} \left[\bigcap_{i \in I} A_i\right],$$

where $[A] : \mathbb{R}^n \to \mathbb{R}$ denotes the indicator function of the set A, $|I|$ is the cardinality of the set I, and the sum in the right-hand side is taken over all non-empty subsets I of $\{1, 2, \dots, \}$

Exercise 2 (Euler characteristic)
Show that the Eulier characteristic can be extended to the space spanned by the indicators $[A]$ of closed convex sets $A \subseteq \mathbb{R}^n$ so that $\chi([A]) = 1$ if A is a non-empty closed convex set.

Exercise 3 (Euler–Poincaré formula)
Let $P \subseteq \mathbb{R}^n$ be a full-dimensional polytope. Show that $[\text{int}(P)] \in \mathscr{P}(\mathbb{R}^n)$ and that $\chi([\text{int}(P)]) = (-1)^n$. Deduce the Euler–Poincaré formula: if P is an n-dimensional polytope, then

$$\sum_{i=1}^{n} (-1)^i f_i = 1,$$

where f_i is the number of i-dimensional faces of P.

Exercise 4 (Polarity)
For a set $X \subseteq \mathbb{R}^n$, the polar X^* of X is the set

$$X^* := \left\{ z \in \mathbb{R}^n : z^{\mathrm{T}} x \le 1 \text{ for all } x \in X \right\}.$$

Show that if P is a polyhedron in $\mathbb{R}^n$ such that $0 \in P$, then

(a) P^* is a polyhedron;

(b) $P^{**} = P$;

(c) $x \in P$ if and only if $\forall z \in P^* : z^{\mathrm{T}} x \le 1$;

(d) if $P = \text{conv}(0, x_1, x_2, \dots, x_m) + \text{cone}(y_1, y_2, \dots, y_k)$, then

$$P^* = \left\{ z \in \mathbb{R}^n : z^{\mathrm{T}} x_i \le 1 \text{ for } i = 1, 2, \dots, m;\ z^{\mathrm{T}} y_j \le 0 \text{ for } j = 1, 2, \dots, k \right\},$$

and conversely. Particularly, if C is a polyhedral cone, then

$$C^* = \left\{ z \in \mathbb{R}^n : z^{\mathrm{T}} x \le 0 \text{ for all } x \in C \right\}.$$