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Location: ELD120 Discussion: 10.03.10

Exercises

Approximation Algorithms

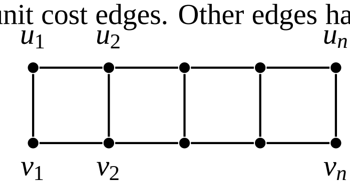
Spring 2010

Sheet 2

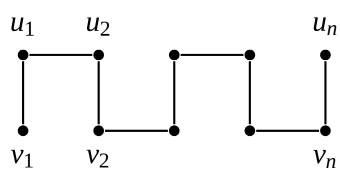
Note: This is just one way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 1
Give a family of instances, where Christophides algorithm for TSP gives a solution whose approximation guarantee indeed tends to $\frac{3}{2}$.

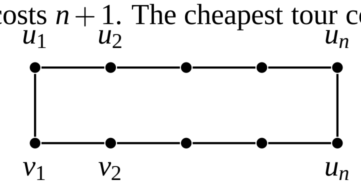
Solution:
Consider a graph with nodes $u_1, \dots, u_n$ on the upper layer, $v_1, \dots, v_n$ on the lower layer. Pairs v_i, u_i and consecutive v_i's, u_i's are connected by unit cost edges. Other edges have shortest path distances.



An MST T of cost $2n-1$ is as follows.



A matching on both odd degree nodes costs $n+1$. The cheapest tour costs $2n$.



Exercise 2
For a parameter $k \in \mathbb{N}$, we consider the following SET COVER instance: Choose elements $U := \mathbb{Z}_2^k \setminus \{(0,\dots,0)\}$. For each vector $z \in \mathbb{Z}_2^k$, we define a set $S_z := \{y \in U \mid z \cdot y \equiv_2 1\}$ where $z \cdot y \equiv_2 \sum_{i=1}^k z_i y_i$ is the standard scalar product mod 2. Hence we have $n := |U| = 2^k - 1$ elements and 2^k sets. All sets have unit cost.

> **Example:** For $k = 2$ we have elements $U = \{(1,0),(0,1),(1,1)\}$ and sets $S_{(0,0)} = \emptyset, S_{(0,1)} = \{(0,1),(1,1)\}, S_{(1,0)} = \{(1,0),(1,1)\}, S_{(1,1)} = \{(1,0),(0,1)\}$.

Show that $OPT \geq k$ and $OPT_f \leq 2$ (hence the integrality gap is $\Omega(\log n)$).

Solution:

We claim that every element is in $\frac{1}{2}2^k \geq \frac{n}{2}$ many sets (i.e. $\frac{1}{2}$ of the sets): Let's fix a $y \in U$. Say $y_i = 1$ ($y \neq (0,\dots,0)$). Fix any choice of $z_1,\dots,z_{i-1},z_{i+1},\dots,z_k \in \{0,1\}$ there is exactly one choice for z_i s.t. $z \cdot y \equiv_2 1$. Hence if we choose $x_i := \frac{2}{n}$, then each element is covered fractionally at least once (in solution x), thus $OPT_f \leq n \cdot \frac{2}{n} = 2$.

Next suppose for contradiction that $k-1$ sets $S_{z^1},\dots,S_{z^{q-1}}$ suffice to cover all elements. Consider the $0/1$ matrix A with rows $z^1,\dots,z^{k-1}$. The rank of this matrix w.r.t. $\mathbb{Z}_2$ can be at most $k-1$. Hence there mut be a non-zero vector $y \in ker(A)$, i.e. $y \cdot z^i \equiv_2 0$ for $i = 1,\dots,k-1$. Hence $y \notin S_{z^i}$.

Exercise 3

The SET PACKING problem is as follows: Given a family of sets $S_1,\dots,S_m \subseteq U$ of cardinality $|S_i| = 3$ with *profits* $c(S_i)$, find a subset of these sets that maximizes the profit, while each element is covered at most once. Consider a straightforward integer linear programming formulation

$$\max \sum_{i=1}^{m} c(S_i) \cdot x_i \qquad (ILP)$$
$$\sum_{i: j \in S_i} x_i \leq 1 \quad \forall j \in U$$
$$x_i \in \{0,1\} \quad \forall i$$

where x_i indicates, whether to take set S_i. Let OPT be its optimum value and OPT_f be the optimum value of its fractional relaxation. Prove that $\frac{OPT_f}{OPT} \leq O(1)$ (for a big enough constant).

Hint: A suitable randomized rounding should do the job.

Solution:

Compute an fractional solution $x^* \in [0,1]^m$ of value OPT_f. Then perform the following rounding algorithm:

(1) Choose set S_i with probability $\frac{1}{6}x_i^*$

(2) Consider all elements $j \in U$: If j is covered by more than 1 set, remove all sets containing j from the solution

Let $I_1 \subseteq \{1,\dots,m\}$ be the sets chosen in (1) and I_2 be the sets chosen in (1) and surviving (2). Consider a set S_i:

$$\Pr[i \in I_2] = \underbrace{\Pr[i \in I_1]}_{=x_i^*} \cdot \Pr[\bigcap_{j:S_j \cap S_i, j \neq i} j \notin I_1] = \frac{1}{6}x_i^* \cdot (1 - \underbrace{\Pr[\bigcup_{j:S_j \cap S_i, j \neq i} j \in I_1]}_{\leq 1/2}) \geq \frac{1}{12}x_i^*$$

Using that

$$\Pr[\bigcup_{j:S_j \cap S_i, j \neq i} j \in I_1] \leq \sum_{j \neq i: S_j \cap S_i} \Pr[j \in I_1] = \frac{1}{6} \underbrace{\sum_{j \neq i: S_j \cap S_i} x_j^*}_{\leq 3} \leq \frac{1}{2}$$

Hence the solution I has an expected profit of

$$\sum_{i=1}^{m} \Pr[i \in I] \cdot c(S_i) \geq \frac{1}{12} \sum_{i=1}^{m} x_i^* c(S_i) = \frac{1}{12} OPT_f$$
