Thomas Rothvoß
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Exercises

Approximation Algorithms

Spring 2010

Sheet 2

Exercise 1
Give a family of instances, where Christophides algorithm for TSP gives a solution whose approximation guarantee indeed tends to $\frac{3}{2}$.

Exercise 2
For a parameter $k \in \mathbb{N}$, we consider the following SET COVER instance: Choose elements $U := \mathbb{Z}_2^k \backslash \{(0,\dots,0)\}$. For each vector $z \in \mathbb{Z}_2^k$, we define a set $S_z := \{y \in U \mid z \cdot y \equiv_2 1\}$ where $z \cdot y \equiv_2 \sum_{i=1}^k z_i y_i$ is the standard scalar product mod 2. Hence we have $n := |U| = 2^k - 1$ elements and 2^k sets. All sets have unit cost.

> **Example:** For $k = 2$ we have elements $U = \{(1,0),(0,1),(1,1)\}$ and sets $S_{(0,0)} = \emptyset, S_{(0,1)} = \{(0,1),(1,1)\}, S_{(1,0)} = \{(1,0),(1,1)\}, S_{(1,1)} = \{(1,0),(0,1)\}$.

Show that $OPT \geq k$ and $OPT_f \leq 2$ (hence the integrality gap is $\Omega(\log n)$).

Exercise 3
The SET PACKING problem is as follows: Given a family of sets $S_1,\dots,S_m \subseteq U$ of cardinality $|S_i| = 3$ with *profits* $c(S_i)$, find a subset of these sets that maximizes the profit, while each element is covered at most once. Consider a straightforward integer linear programming formulation

$$\begin{array}{rcll} \max \sum_{i=1}^m c(S_i) \cdot x_i & & & (ILP) \\ \sum_{i: j \in S_i} x_i & \leq & 1 & \forall j \in U \\ x_i & \in & \{0,1\} & \forall i \end{array}$$

where x_i indicates, whether to take set S_i. Let OPT be its optimum value and OPT_f be the optimum value of its fractional relaxation. Prove that $\frac{OPT_f}{OPT} \leq O(1)$ (for a big enough constant).
Hint: A suitable randomized rounding should do the job.