COMBINATORIAL OPTIMIZATION
PROF. DR. F. EISENBRAND

Final exam

Fall 2011
01.02.2012

Last name: First name:

Exercise:	1	2	3	4	5	6	7	Σ
max points:	8	8	8	8	8	8	8	48
achieved points:								
selected:								

Check whether the exam is complete: it should have 7 pages (Exercises 1–7). Write your name on the title page. Solutions have to be written below the exercises. Solutions must be comprehensible. In case of lack of space, additional paper can be asked from the exam supervision.

You may select 6 out of 7 exercises to work on. Don't forget to mark the chosen exercises!

Use neither pencil nor red colored pen!

Duration: 180 min

Exercise 1: (Multiple Choice, points $\{-1,0,1\}$ each)
No justifications needed. Mark 'yes' or 'no'. **Wrong answers cause negative points!** Total number of points achieved cannot be negative.

a) Let $P \subseteq \mathbb{R}^n$ be a polyhedron, $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. Then $P \cap \{x \in \mathbb{R}^n \mid a^T x = \beta\}$ is a face of P.	○ yes	○ no
b) If $Ax \leq b$ is TDI and $ax \leq \beta$ is a valid inequality for $\{x\colon Ax \leq b\}$, then the system $Ax \leq b, ax \leq \beta$ is also TDI.	○ yes	○ no
c) If $P \neq NP$, then an integer program $\max\{c^T x\colon Ax \leq b, x \in \mathbb{Z}^n\}$ with $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$ can be solved in polynomial time.	○ yes	○ no
d) If $(E, \mathscr{I})$ is a matroid, then for all $A \subseteq E$, every maximal subset $I \subseteq A$ with $I \in \mathscr{I}$ has the same cardinality.	○ yes	○ no
e) Let $G = (V,E)$ be a graph and let $\mathscr{I}$ be the set of all matchings in G. Then $(E, \mathscr{I})$ is a matroid.	○ yes	○ no
f) Let $P \subseteq \mathbb{R}^n$ be a polyhedron with non-empty interior. Then $\mathrm{vol}(P) \geq \frac{1}{n!}$.	○ yes	○ no
g) For a digraph $D = (V,A)$, a set of arcs $B \subseteq A$ is an r-arborescence if and only if B contains no directed cycles and $\lvert B \cap \delta^{in}(v)\rvert = 1$ for all $v \in V \setminus \{r\}$.	○ yes	○ no
h) If $P = NP$, then deciding if a graph contains a spanning tree of certain weight is NP-complete.	○ yes	○ no

Exercise 2: (8 points)
The problem VERTEX COVER is defined as follows: Given a graph $G = (V, E)$, find a set $S \subseteq V$ of minimal cardinality such that every edge $e \in E$ has an endpoint in S.

Give an algorithm that solves this problem in polynomial time on trees.
Prove the correctness of the algorithm and determine its running time.

Solution:

Use reverse side if you need more space

Exercise 3: (8 points)
Let $\mathscr{F} \subseteq \{0,1\}^n$ and assume access to an oracle that given $\bar{x} \in \mathscr{F}$ and $\bar{c} \in \mathbb{Z}^n$ either

- asserts that $\bar{x} \in \mathscr{F}$ maximizes $\bar{c}^T x$ over $x \in \mathscr{F}$, or
- returns an $x \in \mathscr{F}$ such that $\bar{c}^T x > \bar{c}^T \bar{x}$.

Perform the following tasks:

1. Suppose $c = 2\tilde{c} + v$, where $c, \tilde{c} \in \mathbb{Z}^n$ and $v \in \{0, \pm 1\}^n$. Suppose you know the solution $\tilde{x} \in \mathscr{F}$ that attains $\max\{\tilde{c}^T x \mid x \in \mathscr{F}\}$. How many oracle calls do you need at most to find the optimum solution with respect to c?
2. Describe an algorithm that given $c \in \mathbb{Z}^n$ and an initial feasible solution $x \in \mathscr{F}$ computes an optimal solution of $\max\{c^T x \mid x \in \mathscr{F}\}$ with a running time that is bounded by a polynomial in n and $\log |c|_\infty$. Prove the correctness of your answer.

Solution:

Use reverse side if you need more space

Exercise 4: (8 points)
Let $G = (V,E)$ be a complete(!) graph. A *2-matching* is an assignment $x : E \to \{0,1,2\}$ such that for each vertex $v \in V$: $\sum_{e\in\delta(v)} x(e) = 2$, where $\delta(v)$ denotes the set of edges incident at v. Informally, a 2-matching is a collection of disjoint cycles and some edges (that are taken twice).

The problem MINIMUM WEIGHT 2-MATCHING is defined as follows: Given the complete graph $G = (V,E)$, a weight function on the edges $w : E \to \mathbb{R}^+$, find a 2-matching that minimizes $\sum_{e\in E} w(e)x(e)$.

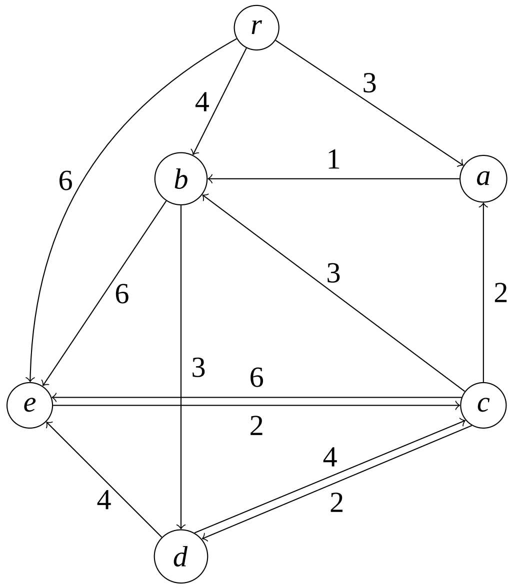
Give an algorithm that solves this problem in polynomial time.

Hint: Create a new vertex v' for each vertex v of G, and replace each edge vw of G by two edges $\{v',w\}$ and $\{v,w'\}$. Continue using matching techniques.

Solution:

Use reverse side if you need more space

Exercise 5: (8 points)
Compute a minimum weight arborescence rooted at r in the following digraph.



Prove the optimality of your solution!

Solution:

Use reverse side if you need more space

Exercise 6: (8 points)
Let $G = (V,E)$ be an undirected graph and let M_1, $M_2 \subseteq E$ be matchings with $|M_2| > |M_1|$. Let $V(M_1)$ and $V(M_2)$ denote the vertices covered by M_1 and M_2, respectively. Prove that one can augment M_1 in the following sense: There exists a matching $M' \subseteq E$ such that $V(M_1) \subsetneq V(M') \subseteq V(M_1) \cup V(M_2)$.

Solution:

Use reverse side if you need more space

Exercise 7: (8 points)
Consider the following problem:

> k-MST
> *Given:* an undirected graph $G = (V, E)$ with weights $c \colon E \to R_+$ and numbers $k, M \in \mathbb{N}$
> *Task:* Is there a tree $T = (V', E')$ (with $V' \subseteq V$ and $E' \subseteq E$) of weight at most M that spans at least k vertices (i.e. $|V'| \geq k$)?

Show that

1. k-MST is in NP.
2. k-MST is NP-complete.

Hints:

- Consider :

  > STEINER TREE PROBLEM
  > *Given:* an undirected graph $G = (V, E)$, a set of terminals $R \subseteq V$ and $M \in \mathbb{N}$
  > *Task:* Is there a tree $T = (V', E')$ (with $V' \subseteq V$ and $E' \subseteq E$) of at most M edges that spans all vertices in R ?

- Use the following construction: Connect each terminal of G to a distinct path of $|V|$ new vertices, consisting of zero-weighted edges. Assign weight 1 to the already existing edges in G (and set the weight of all other pairs to ∞). Set $k = |R| \cdot |V|$.

Solution:

Use reverse side if you need more space