DISCRETE OPTIMIZATION
PROF. F. EISENBRAND

Exam

Spring 2010
01.07.2010

Last name: First name:

Exercise:	1	2	3	4	5	6	7	8	Σ
max points:	10	10	10	10	10	10	10	10	50
achieved points:									
chosen exercises:									

Check whether the exam is complete: It should have 12 pages (Exercises 1–8). Write your name on the title page. Solutions have to be written below the exercises. Solutions must be comprehensible. In case of lack of space, you can ask for additional paper from the exam supervision. Please put your name on each additional sheet and indicate which exercise it belongs to.

Use neither pencil nor red colored pen!

Duration: 180 min

Grading:
Every exercise gives 10 points, and you are supposed to solve 5 of them. There are 6 exercises marked with [$*$] and two exercises marked with [Δ]. Math students can choose among the [$*$]-exercises. Non-math students can choose among all exercises. **Please mark the 5 exercises you have chosen in the tabular above!**

You are allowed to bring a pocket calculator and an A4-"cheat-sheet".

Exercise 1 [$*$]: (Multiple Choice, points $\{-1, 0, 1\}$ each)
No justifications needed. Mark 'yes' or 'no'. **Wrong answers cause negative points!**

a) Given a set $A := \{a_1, \dots, a_m\} \subset \mathbb{R}^n$, then for all $x \in \text{conv}(A)$ there are $\lambda_1, \dots, \lambda_m \in \mathbb{R}_{\geq 0}$, at most n of them nonzero, such that $\sum_{i=1}^m \lambda_i = 1$ and $x = \sum_{i=1}^m \lambda_i a_i$.	○ yes	○ **no**
b) Given a linear program $\max\{c^T x \mid Ax \leq b\}$, with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ and $m > n$. If x is an optimal solution, at least n of the LPs inequalities are satisfied with equality by x.	○ yes	○ **no**
c) Given a linear program $\max\{c^T x \mid Ax \leq b\}$, with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. If the LP is feasible and bounded, then there is a *roof* B such that its vertex is an optimal solution of the LP.	○ yes	○ **no**
d) For a linear program $\max\{c^T x \mid Ax \leq b\}$, with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, let j be a row that entered the roof in the ith iteration of the simplex algorithm. Then it cannot leave the roof in the $(i+1)$st iteration.	○ yes	○ **no**
e) For a linear program $\max\{c^T x \mid Ax \leq b\}$, with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, let j be a row that has left the roof in the ith iteration of the simplex algorithm. Then it cannot reenter into the roof in the $(i+1)$st iteration.	○ **yes**	○ no
f) Given a linear program $\max\{c^T x \mid Ax \leq b\}$, with $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m, c \in \mathbb{R}^n$, A full column rank. If A is totally unimodular, then the vertex of every roof of the LP is an integer vector.	○ **yes**	○ no
g) Consider an integer program $\max\{c^T x \mid Ax \leq b,\ x \in \mathbb{Z}^n\}$, with $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m, c \in \mathbb{R}^n$. Given an optimal solution x^* to the IP, there is a solution y^* to the IP $\min\{b^T y \mid A^T y = c,\ y \in \mathbb{Z}^m_{\geq 0}\}$ such that $c^T x^* = b^T y^*$.	○ yes	○ **no**
h) There is a linear program $\max\{c^T x \ :\ Ax \leq b\}$, with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ such that both the LP and its dual are infeasible.	○ **yes**	○ no
i) Given a directed graph $G = (V, A)$ and a length function $\ell : A \to \mathbb{Z}$. One can decide if a negative cycle in G (w.r.t. length ℓ) exists in time $O(\lvert V \rvert \cdot \lvert A \rvert)$.	○ **yes**	○ no
j) Given a directed graph $G = (V, A)$ and two nodes $s, t \in V$. The maximum number of arc-disjoint $s-t$-paths is equal to the smallest number of arcs in an $s-t$-cut. [A collection $P_1, \dots, P_k$ is arc-disjoint, if no pair of paths has an arc in common.]	○ **yes**	○ no

Exercise 2 [Δ] (LP modelling 1):

A refinery can produce gas of two types of qualities A and B, composed of three kinds of crude oil 1, 2, 3. To produce one unit of gas A or B, one can combine fractional quantities of the three kinds of crude oils such that they sum up to 1. The following restrictions apply: At most 30% of crude oil 1, at least 40% of crude oil 2 and at most 50% of crude oil 3 has to be used to produce gas A. At most 50% of crude oil 1 and at least 10% of cruide oil 2 has to be used to produce gas B (There are no restrictions for crude oil 3 here). The total quantity of each crude oil that can be used in production is limited, and it comes with a cost per unit which is as follows:

crude oil	max quantity	cost per unit
1	3000	3
2	2000	6
3	4000	4

One unit of gas A sells for 5.5, and one unit of gas B sells for 4.5.
Formulate the problem of maximizing revenue (income by selling the gas minus cost of raw material) as a linear program.
You do not need to transform you LP into standard form!

Solution:
We introduce variables as follows:

- x_A: Amount of gas A to be produced.
- x_B: Amount of gas B to be produced.
- y_{ij}: Amount of crude i being used to produce gas j ($i = 1,2,3,\ j = A,B$).

The LP looks as follows:

$$\begin{aligned}
\max \quad & 5.5 \cdot x_A + 4.5 \cdot x_B - 3 \cdot (y_{1A} + y_{1B}) - 6 \cdot (y_{2A} + y_{2B}) - 4 \cdot (y_{3A} + y_{3B}) \\
\text{s.t.} \quad & x_A = y_{1A} + y_{2A} + y_{3A} \\
& x_B = y_{1B} + y_{2B} + y_{3B} \\
& y_{1A} \leq 0.3 x_A \\
& y_{2A} \geq 0.4 x_A \\
& y_{3A} \leq 0.5 x_A \\
& y_{1B} \leq 0.5 x_B \\
& y_{2B} \geq 0.1 x_B \\
& y_{1A} + y_{1B} \leq 3000 \\
& y_{2A} + y_{2B} \leq 2000 \\
& y_{3A} + y_{3B} \leq 4000
\end{aligned}$$

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Exercise 3 [$*$] (LP modelling 2):

Consider the polyhedron

$$P := \{x \in \mathbb{R}^n \ : \ Ax \leq b\}$$

for some $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$.
A *ball* with center y and radius r is defined as the set:

$$B_r(y) := \{x \in \mathbb{R}^n \ : \ ||x-y||_2 \leq r\}.$$

We want to find a ball (i.e. its center y) of largest possible radius which is entirely contained in P.
Formulate this problem as a linear program. **Prove that your LP formulation gives the desired result!**
Hints:

- *For* $v \in \mathbb{R}^n$, $||v||_2{}^2 = v^T v$.
- *Consider each row of* $a_i x \leq b_i$ *individually. Find a necessary and sufficient condition that asserts that* $B_r(y)$ *does not violate* $a_i x \leq b_i$.

Solution:
We introduce a variable r to model the radius and a vector y to model the center of the ball. The following LP solves the problem:

$$\begin{aligned} \max \quad & r \\ \text{s.t.} \quad & a_i^T y + r \cdot ||a_i||_2 \leq b_i \quad \forall i \end{aligned}$$

To show correctness, we show that each "feasible" ball of radius r gives rise to a feasible solution of the LP (with objective value r) and vice versa.
First let $B_r(y)$ be a ball such that $B_r(y) \subseteq P$. We need to show that r, y is a feasible solution to the LP: Fix a constraint i. Observe that $\left|\left|y + r \cdot \frac{a_i}{||a_i||_2}\right|\right|_2 \in B_r(y)$ because

$$\left|\left|y + r \cdot \frac{a_i}{||a_i||_2} - y\right|\right|_2 = r.$$

Thus $y + r \cdot \frac{a_i}{||a_i||_2} \in P$, i.e.

$$b_i \geq a_i^T (y + r \cdot \frac{a_i}{||a_i||_2}) = a_i^T y + r||a_i||_2.$$

This shows that r, y is a feasible solution to the LP.
Now consider a feasible solution r, y to the LP. We need to show that $B_r(y) \subseteq P$ holds. Thus let $x \in B_r(y)$ Fix some constraint i. We can write $x - y = \alpha \cdot a_i + \beta \cdot v$, where v is a vector orthogonal to a_i, i.e. $a_i^T v = 0$.
Moreover $r^2 \geq ||x-y||_2{}^2 = \alpha^2 \cdot ||a_i||_2{}^2 + \beta^2 \cdot ||v||_2{}^2$ (Pythagoras). Hence $\alpha \leq \frac{r}{||a_i||_2}$.
Finally

$$\begin{aligned} a_i^T x &= a_i^T (y + x - y) \\ &= a_i^T \left(y + a_i^T (\alpha \cdot a_i + \beta \cdot v)\right) \\ &= a_i^T y + \alpha ||a_i||_2{}^2 \\ &\leq a_i^T y + r||a_i||_2 \leq b_i. \end{aligned}$$

This shows that $x \in P$.

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Exercise 4 [$*$] (LP Duality / Complementary slackness):

(a) Consider a linear program

$$\max\{c^T x \ : \ Ax \leq b\}$$

for some $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$.

Prove the complementary slackness theorem: Given an optimal solution x^* for the LP, there is an optimal solution y^* for its dual such that $y_i^* = 0$ for all rows i of the primal that are not satisfied with equality by x^*.

Now consider the following LP:

$$\begin{array}{lcccccc} \max & 10x_1 & + & 4x_2 & & \\ & x_1 & & & \leq & -4 \\ & & & x_2 & \leq & -6 \\ & x_1 & + & 2x_2 & \leq & -20 \\ & 2x_1 & + & x_2 & \leq & -17 \end{array}$$

(b) Write down a dual of the LP in standard form.

(c) Show that $x^* := (-4, -9)^T$ is an optimal solution for the LP by giving a suitable solution for the dual LP (using the complementary slackness theorem).

Solution:

(a) $0 = c^T x - b^T y = \sum y^T Ax - b^T y = y^T(Ax - b)$. Each component of the two vectors involved in that scalar product are nonnegative. Claim follows.

(b) A dual is:

$$\begin{array}{lcccccccc} \min & -4y_1 & - & 6y_2 & - & 20y_3 & - & 17y_4 & & \\ & y_1 & & + & y_3 & + & 2y_4 & = & 10 \\ & & & y_2 & 2y_3 & + & y_4 & = & 4 \end{array}$$

(c) With complementary slackness we get that $y_2 = y_3 = 0$. Hence the second constraint yields $y_4 = 4$. Thus $y_1 = 2$. Our dual solution is $y = (2, 0, 0, 4)$. Both the primal and the dual solution have a value of -76, which asserts that they are optimal.

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Solution:

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Exercise 5 [$*$] (Roofs):

Consider a linear program

$$\max\{c^T x \;:\; Ax \leq b\}$$

for some $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ where A has full column rank.
Consider a subset $B \subseteq \{1,\ldots,m\}$ of rows such that $|B| = n$ and the rows a_i, $i \in B$ are linearly independent.
Show that if $c \in \text{cone}\{a_i \;:\; i \in B\}$, then B is a roof.

Solution:
See proof of Lemma 3.2 in lecture notes.

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Exercise 6 [$*$] (Simplex algorithm):

Consider the following LP:

$$
\begin{array}{rrcrcrcl}
\max & 2y_1 & + & y_2 & + & y_3 & & \\
 & y_1 & + & 2y_2 & & & \leq & -3 \\
 & -2y_1 & - & 3y_2 & & & \leq & 5 \\
 & -2y_1 & - & y_2 & + & 2y_3 & \leq & -1 \\
 & y_1 & & & & & \leq & 0 \\
 & y_2 & & & & & \leq & 0 \\
 & y_3 & & & & & \leq & 0
\end{array}
$$

Solve the LP using the simplex method.
Start with the roof $B = \{4,5,6\}$. **In the first iteration, let row 1 enter the roof!**
For each iteration of the simplex method, specify the row that should enter the roof, the row that has to leave the roof, the new roof and its vertex.

Also write down an optimal solution and its value.

These are the inverse matrices for all roofs:

$$B := \{1,3,4\}, A_B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}. \quad B := \{1,3,5\}, A_B^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 1 & \frac{1}{2} & -\frac{3}{2} \end{pmatrix}.$$

$$B := \{1,4,6\}, A_B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad B := \{3,4,5\}, A_B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}.$$

$$B := \{4,5,6\}, A_B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution:
Let A be the matrix of the LP. We start with the roof $B = \{4,5,6\}$. The corresponding vertex solution is $y = (0,0,0)^T$. The conic combination of c is $c = \mu_1 \cdot a_4 + \mu_2 \cdot a_5 + \mu_3 \cdot a_6$ with $\mu = (2,1,1)$.
Both constraints 1 and 3 are violated. We choose 1 to enter the roof. Hence we compute a solution to the system

$$\lambda_1 a_4 + \lambda_2 a_5 + \lambda_3 a_6 + a_1 = 0,$$

which is $\lambda = (-1,-2,0)$. We have $-\frac{\mu_1}{\lambda_1} = 2$ and $-\frac{\mu_2}{\lambda_2} = \frac{1}{2}$, thus 5 leaves the roof. Hence we get the new conic combination $c = \mu_1 \cdot a_1 + \mu_2 \cdot a_4 + \mu_3 \cdot a_6$ with $\mu = (\frac{1}{2}, \frac{3}{2}, 1)$ and the new roof is $B = \{1,4,6\}$ with vertex solution $y = (0, -\frac{3}{2}, 0)^T$.
Constraint 3 is violated and enters the roof. Hence we compute a solution to the system

$$\lambda_1 a_1 + \lambda_2 a_4 + \lambda_3 a_6 + a_3 = 0,$$

which is $\lambda = (\frac{1}{2}, \frac{3}{2}, -2)$. We have $-\frac{\mu_3}{\lambda_3} = \frac{1}{2}$ thus 6 leaves the roof. Hence we get the new conic combination $c = \mu_1 \cdot a_1 + \mu_2 \cdot a_3 + \mu_3 \cdot a_4$ with $\mu = (\frac{3}{4}, \frac{9}{4}, \frac{1}{2})$ and the new roof is $B = \{1,3,4\}$ with vertex solution $y = (0, -\frac{3}{2}, -\frac{1}{4})^T$.
This solution satisfies all constraints, thus it is optimal. The objective value is $-\frac{3}{2} - \frac{1}{4} = -\frac{7}{4}$.

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Solution:

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Exercise 7 [$*$] (Flow modeling):

There are k projects available to be undertaken. Let P be the set of projects. Each project $i \in P$ has associated a positive revenue $r_i \in \mathbb{N}$ and requires a set $S_i \subseteq R$ of resources to be available. Each resource $j \in R$ has an associated cost $c_j \in \mathbb{N}$.
If a resource $j \in R$ is required by several projects, it needs to be bought only once.
Show how to use a max-flow (min-cut) algorithm to solve the problem of choosing a subset of projects, such that the sum of the revenues minus the cost of the required resources is maximized.
Hint: Construct a graph such that every cut of finite capacity corresponds to a set of selected projects and all their required resources. (In particular, it might be useful to put a capacity of ∞ on some arcs.)

Solution:
Construct the following graph $G = (V, A)$:
$V := P \cup R + s, t$. $A := \{(i, j) \ : \ i \in P, \ j \in S_i\} \cup \{(s, i) \ : \ i \in P\} \cup \{(j, t) \ : \ j \in R\}$. We define capacities

$$u : A \to \mathbb{N} \cup \{\infty\}, \ a \mapsto \begin{cases} r_i, & \text{if } a = (s, i), \, i \in P \\ c_j, & \text{if } a = (j, t), \, j \in P. \\ \infty, & \text{else} \end{cases}$$

Note that every finite capacity $s-t$-cut in G corresponds to a selection of projects and their required resources in a natural way: The projects "on the same side" of the cut are the ones that are selected. By construction, all required resources are on the "same side" as well (otherwise the cut would have infinite capacity). Conversely, one can transform each selection of projects to a finite capacity cut.
Let P' be the set of projects selected, and R' be the set of projects required by P'. Then The capacity of the cut is given as

$$\sum_{i \in P \setminus P'} u(s, i) + \sum_{j \in R'} u(j, t) = \sum_{i \in P \setminus P'} r_i + \sum_{j \in R'} c_j = \sum_{i \in P} r_i - (\sum_{i \in P'} r_i - \sum_{j \in R'} c_j).$$

This shows that a selection of projects of maximum revenue can be found by computing a min capacity $s-t$-cut in G.

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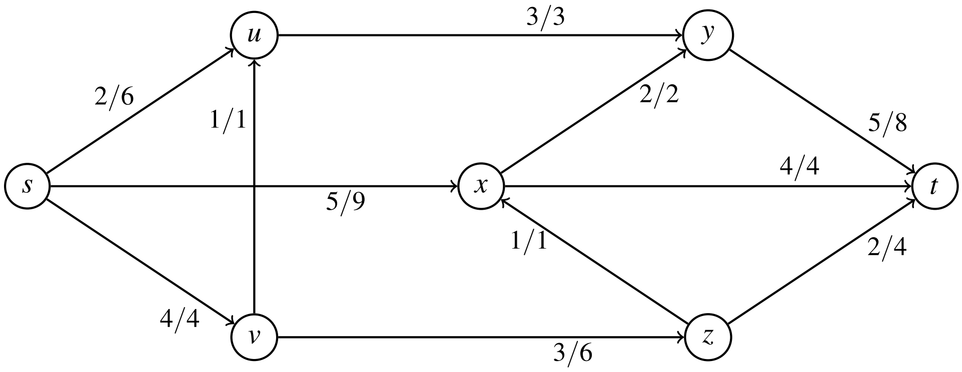
Exercise 8 [Δ] (Max $s-t$-flows):

Consider the following graph $G=(V,A)$. The labels on the arcs $a \in A$ are of the form $f(a)/u(a)$, i.e. they define functions $f:\ A \to \mathbb{Q}_{\geq 0}$ and $u:\ A \to \mathbb{Q}_{\geq 0}$.



(a) Argue why f is a *feasible* $s-t$*-flow* in G subject to the capacities u. What is the value of the flow?

(b) Perform the Ford-Fulkerson algorithm to compute a maximum $s-t$-flow in G. For each iteration give the residual network. You can start with the flow f. Give the flow, its value and a minimum $s-t$ cut.

(c) Prove the weak duality theorem: Given a feasible $s-t$-flow f and an $s-t$-cut $\delta^{out}(U)$ for some $U \subset V$, then

$$\text{value}(f) \leq u(\delta^{out}(U).$$

Hint: You can use the fact that $\text{excess}_f(U) = \sum_{v \in U} \text{excess}_f(v)$ *for each* $U \subseteq V$.

Solution:

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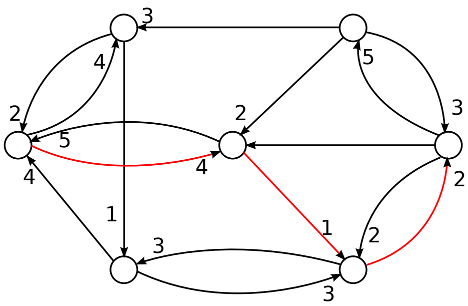
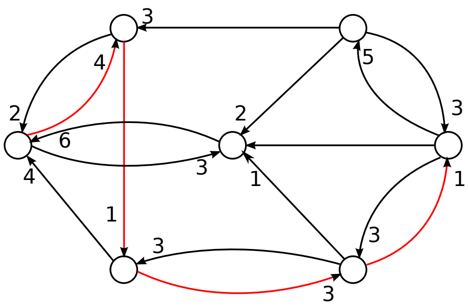
Solution:

(a) f is a *flow* since for all $v \in V - s,t$ we have $f(\delta^{in}(v)) = f(\delta^{in}(v))$. f is feasible since $0 \leq f(a) \leq u(a)$ for each $a \in A$.

(b) We start with the residual network corresponding to f.



An $s-t$-path is marked red. After augmenting we obtain the following residual network:



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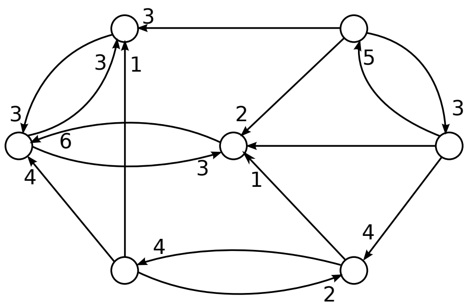

s and t are not connected anymore. Thus we found the optimal $s-t$-flow f given as

$$f(s,u) = 3,\ f(s,v) = 4,\ f(s,x) = 6,\ f(u,y) = 3,\ f(v,u) = 0,\ f(v,z) = 4,$$

$$f(x,y) = 2,\ f(x,z) = 0,\ f(x,t) = 4,\ f(y,t) = 5,\ f(z,t) = 4.$$

of value 13. A min cut is given by the nodes $\{s,u,x\}$.

(c) $\text{value}(f) = -\text{excess}_f(s) = -\sum_{v \in U} \text{excess}_f(v) = -\text{excess}_f(U) = f(\delta^{out}(U)) - f(\delta^{in}(U)) \leq u(\delta^{out}(U))$.

Use reverse side if you need more space